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Stability, Existence and Uniqueness of Boundary Value Problems for a Coupled System of Fractional Differential Equations

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Abstract: The current article studies a coupled system of fractional differential equations with boundary conditions and proves the existence and uniqueness of solutions by applying Leray-Schauder's alternative and contraction mapping principle. Furthermore, the Hyers-Ulam stability of solutions is discussed and sufficient conditions for the stability are developed. Obtained results are supported by examples and illustrated in the last section.

Keywords: fractional derivative; fixed point theorem; fractional differential equation

1. Introduction

Fractional calculus is undoubtedly one of the very fast-growing fields of modern mathematics, due to its broad range of applications in various fields of science and its unique efficiency in modeling complex phenomena [1,2]. In particular, fractional differential equations with boundary conditions are widely employed to build complex mathematical models for numerous real-life problems such as blood flow problem, underground water flow, population dynamics, and bioengineering. As an example, consider the following equation that describes a thermostat model

$$-x'' = g(t)f(t,x), x(0) = 0, \beta x\prime(1) = x(\eta),$$

where $t \in (0,1), \eta \in (0,1]$ and β is a positive constant. Note that solutions of the above equation with the specified integral boundary conditions are in fact solutions of the one-dimensional heat equation describing a heated bar with a controller at point 1, which increases or reduces heat based on the temperature picked by a sensor at η. A few of the relevant studies on coupled systems of fractional differential equations with integral boundary conditions are briefly reviewed below and for further information on this topic, refer to References [3,4].

In Reference [5], Ntouyas and Obaid used Leray-Schauder's alternative and Banach's fixed-point theorem to prove the existence and uniqueness of solutions for the following coupled fractional differential equations with Riemann-Liouville integral boundary conditions:

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u(t) = g(t,u(t),v(t)), t \in [0,1], \\ {}^{c}D^{\beta}_{0+}v(t) = g(t,u(t),v(t)), t \in [0,1], \\ u(0) = \gamma I^{p}u(\eta) = \gamma \int_0^{\eta} \frac{(\eta-s)^{p-1}}{\Gamma(p)} u(s)ds, 0 < \eta < 1, \\ v(0) = \delta I^{q}v(\zeta) = \delta \int_0^{\zeta} \frac{(\zeta-s)^{q-1}}{\Gamma(q)} v(s)ds, 0 < \zeta < 1. \end{cases}$$

Here, ${}^cD^{\alpha}_{0+}$ and ${}^cD^{\beta}_{0+}$ are Caputo fractional derivatives, $0 < \alpha, \beta \le 1$, $f, g \in C\Big([0,1] \times \mathbb{R}^2, \mathbb{R}\Big)$ and $p, q, \gamma, \delta \in \mathbb{R}$.

Similarly, Ahmed and Ntouyas [6] employed Banach fixed-point theorem and Leray-Schauder's alternative to prove the existence and uniqueness of solutions for the following coupled fractional differential system:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), y(t)), & t \in [0,1], \quad 1 < q \le 2, \\ {}^cD^p y(t) = g(t, x(t), y(t)), & t \in [0,1], \quad 1 < q \le 2, \end{cases}$$

supplemented with coupled and uncoupled slit-strips-type integral boundary conditions, respectively, given by

$$\begin{cases} x(0) = 0, \quad x(\zeta) = a\int_0^{\eta} y(s)ds + b\int_{\xi}^1 y(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \quad y(\zeta) = a\int_0^{\eta} x(s)ds + b\int_{\xi}^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \end{cases}$$

and

$$\begin{cases} x(0) = 0, \quad x(\zeta) = a\int_0^{\eta} x(s)ds + b\int_{\xi}^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \quad y(\zeta) = a\int_0^{\eta} y(s)ds + b\int_{\xi}^1 y(s)ds, \quad 0 < \eta < \zeta < \xi < 1. \end{cases}$$

Furthermore, Alsulami et al. [7] investigated the following coupled system of fractional differential equations:

$$\begin{cases} {}^cD^{\alpha} x(t) = f(t, x(t), y(t)), t \in [0,T], 1 < \alpha \le 2, \\ {}^cD^{\beta} y(t) = g(t, x(t), y(t)), t \in [0,T], 1 < \beta \le 2, \end{cases}$$

subject to the following non-separated coupled boundary conditions:

$$\begin{cases} x(0) = \lambda_1 y(T), x'(0) = \lambda_2 y'(T), \\ y(0) = \mu_1 x(T), y'(0) = \mu_2 x'(T). \end{cases}$$

Note that ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ denote Caputo fractional derivatives of order α and β. Moreover, λ_i, μ_i, $i = 1,\ 2$, are real constants with $\lambda_i\mu_i \neq 1$ and $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are appropriately chosen functions. For further details on this topic, refer to References [8–21].

The current paper studies the following coupled system of nonlinear fractional differential equations:

$$\begin{cases} {}^cD^{\alpha} x(t) = f(t, x(t), y(t)), \quad t \in [0,T], \quad 1 < \alpha \le 2, \\ \\ {}^cD^{\beta} y(t) = g(t, x(t), y(t)), \quad t \in [0,T], \quad 1 < \beta \le 2, \end{cases} \tag{1}$$

supplemented with boundary conditions of the form:

$$x(T) = \eta y'(\rho), \quad y(T) = \zeta x\prime(\mu), \quad x(0) = 0, \quad y(0) = 0, \rho, \mu \in [0,T] \tag{2}$$

Here, ${}^cD^k$ denotes Caputo fractional derivative of order k $(k = \alpha, \beta)$; and $f, g \in C\Big([0,T] \times \mathbb{R}^2, \mathbb{R}\Big)$ are given continuous functions. Note that η, ζ are real constants such that $T^2 - \eta\zeta \neq 0$.

The rest of this paper is organized in the following manner: In Section 2, we briefly review some of the relevant definitions from fractional calculus and prove an auxiliary lemma that will be used later. Section 3 deals with proving the existence and uniqueness of solutions for the given problem, and Section 4 discusses the Hyers-Ulam stability of solutions and presents sufficient conditions for the stability. The paper concludes with supporting examples and obtained results.

2. Preliminaries

We begin this section by reviewing the definitions of fractional derivative and integral [1,2].

Definition 1. *The Riemann-Liouville fractional integral of order τ for a continuous function h is given by*

$$I^{\tau}h(s) = \frac{1}{\Gamma(\tau)}\int_0^s \frac{h(t)}{(s-t)^{1-\tau}}dt, \quad \tau > 0,$$

provided that the right-hand side is point-wise defined on $[0, \infty)$.

Definition 2. *The Caputo fractional derivatives of order τ for $(h-1)$—times absolutely continuous function $g : [0, \infty) \to \mathbb{R}$ is defined as*

$${}^{c}D^{\tau}g(s) = \frac{1}{\Gamma(h-\tau)}\int_0^s (s-t)^{h-\tau-1}g^{(h)}(t)dt, \quad h-1 < \tau < h, \quad h = [\tau]+1,$$

where $[\tau]$ is the integer part of real number τ.

Here we prove the following auxiliary lemma that will be used in the next section.

Lemma 1. *Let $u, v \in C([0,T], \mathbb{R})$ then the unique solution for the problem*

$$\begin{cases} {}^{c}D^{\alpha}x(t) = u(t), \quad t \in [0,T], \quad 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}y(t) = v(t), \quad t \in [0,T], \quad 1 < \beta \leq 2, \\ x(T) = \eta y'(\rho), \quad y(T) = \zeta x'(\mu), \quad x(0) = 0, \quad y(0) = 0, \rho, \mu \in [0,T] \end{cases} \tag{3}$$

is

$$\begin{aligned} x(t) = \; & \tfrac{t}{\Delta}\Big(\eta T \int_0^{\rho} \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds - T\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds + \eta\zeta\int_0^{\mu}\tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds - \eta\int_0^T\tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\Big) \\ & + \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds, \end{aligned} \tag{4}$$

and

$$\begin{aligned} y(t) = \; & \tfrac{t}{\Delta}\Big(\eta\zeta \int_0^{\rho} \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds - \zeta\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds + T\zeta\int_0^{\mu}\tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s) - T\int_0^T\tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\Big) \\ & + \int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds \end{aligned} \tag{5}$$

where $\Delta = T^2 - \eta\zeta \neq 0$.

Proof. General solutions of the fractional differential equations in (3) are known [6] as

$$\begin{aligned} x(t) &= at + b + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \\ y(t) &= ct + d + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds, \end{aligned} \tag{6}$$

where $a, b, c,$ and d are arbitrary constants.

Apply conditions $x(0) = 0$ and $y(0) = 0$, and we obtain $b = d = 0$.

Here

$$x'(t) = a + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}u(s)ds,$$

$$y'(t) = c + \frac{1}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2}v(s)ds.$$

Considering boundary conditions

$$x(T) = \eta y'(\rho), \quad y(T) = \zeta x'(\mu)$$

we get

$$aT+\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds=\eta c+\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds,$$

and

$$cT+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds=a\zeta+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds,$$

so

$$a=\frac{1}{T}\left(\eta c+\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\right),$$
$$c=\frac{1}{T}\left(a\zeta+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right).$$

Hence, by substituting the value of a into c, we obtain the final result for these constants as

$$c=\tfrac{1}{T}\left(\tfrac{\zeta}{T}\left[\eta c+\eta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\right]+\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

$$c-\tfrac{\zeta\eta c}{T^2}=\tfrac{1}{T}\left(\tfrac{\zeta}{T}\left[\eta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\right]+\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

$$c\left(\tfrac{T^2-\zeta\eta}{T^2}\right)=\tfrac{1}{T}\left(\tfrac{\zeta}{T}\left[\eta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\right]+\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

$$c=\tfrac{T}{T^2-\zeta\eta}\left(\tfrac{\zeta}{T}\left[\eta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\right]+\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

$$c=\tfrac{1}{T^2-\zeta\eta}\left(\eta\zeta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\zeta\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+T\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-T\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right)$$

$$c=\tfrac{1}{\Delta}\left(\eta\zeta\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\zeta\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+T\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-T\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

and

$$a=\tfrac{1}{\Delta}\left(\eta T\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-T\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+\eta\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds-\eta\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\right),$$

Substituting the values of $a, b, c,$ and d in (6) and (7) we get (4) and (5). The converse follows by direct computation. This completes the proof. □

3. Existence and Uniqueness of Solutions

Consider the space $C([0,T],\mathbb{R})$ endowed with norm $\|x\| = \sup\limits_{0\le t\le T}\left|x(t)\right|$. Consequently, the product space $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ is a Banach Space (endowed with $\|(x,y)\|=\|x\|+\|y\|$).

In view of Lemma 1, we define the operator $G: C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ as:

$$G(x,y)(t)=(G_1(x,y)(t),G_2(x,y)(t)),$$

where

$$\begin{aligned}G_1(x,y)(t)=\tfrac{t}{\Delta}\quad&\left(\eta T\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}g(s,x(s),y(s))ds-T\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),y(s))ds\right.\\&\left.+\eta\zeta\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s),y(s))ds-\eta\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s,x(s),y(s))ds\right)\\&+\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),y(s))ds,\end{aligned}\tag{7}$$

and

$$\begin{aligned} G_2(x,y)(t) = \tfrac{t}{\Delta} \quad & \Big(\eta\zeta \int_0^{\rho} \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} g(s,x(s),y(s))ds - \zeta \int_0^{T} \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),y(s))ds \\ & +T\zeta \int_0^{\mu} \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s),y(s))ds - T \int_0^{T} \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s),y(s))ds\Big) \\ & + \int_0^{t} \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s),y(s))ds, \end{aligned} \tag{8}$$

Here we establish the existence of the solutions for the boundary value problem (1) and (2) by using Banach's contraction mapping principle.

Theorem 1. *Assume $f, g : C([0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are jointly continuous functions and there exist constants $\phi, \psi \in \mathbb{R}$, such that $\forall\ x_1, x_2, y_1, y_2 \in \mathbb{R}, \forall t \in [0,T]$, we have*

$$\big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| \leq \phi\Big(|x_2 - x_1| + \big|y_2 - y_1\big|\Big),$$

$$\big|g(t,x_1,x_2) - f(t,y_1,y_2)\big| \leq \psi\Big(|x_2 - x_1| + \big|y_2 - y_1\big|\Big),$$

where

$$\phi(Q_1 + Q_3) + \psi(Q_2 + Q_4) < 1,$$

then the BVP (1) and (2) has a unique solution on $[0,T]$. Here

$$\begin{aligned} Q_1 &= \tfrac{T}{|\Delta|}\Big(\tfrac{T^{\alpha+1}}{\Gamma(\alpha+1)} + \tfrac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\Big) + \tfrac{T^{\alpha}}{\Gamma(\alpha+1)}, \\ Q_2 &= \tfrac{T}{|\Delta|}\Big(\tfrac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \tfrac{|\eta|T^{\beta}}{\Gamma(\beta+1)}\Big), \\ Q_3 &= \tfrac{T}{|\Delta|}\Big(\tfrac{|\zeta|T^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\Big), \\ Q_4 &= \tfrac{T}{|\Delta|}\Big(\tfrac{|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)} + \tfrac{T^{\beta+1}}{\Gamma(\beta+1)}\Big) + \tfrac{T^{\beta}}{\Gamma(\beta+1)}. \end{aligned} \tag{9}$$

Proof. Define $\underset{0 \leq t \leq T}{sup} \big|f(t,0,0)\big| = f_0 < \infty$, $\underset{0 \leq t \leq T}{sup} \big|g(t,0,0)\big| = g_0 < \infty$ and $\Omega_\varepsilon = \{(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) : \|(x,y)\| \leq \varepsilon\}$, and $\varepsilon > 0$, such that

$$\varepsilon \geq \frac{(Q_1 + Q_3)f_0 + (Q_2 + Q_4)g_0}{1 - [\phi(Q_1 + Q_3) + \psi(Q_2 + Q_4)]}.$$

Firstly, we show that $G\Omega_\varepsilon \subseteq \Omega_\varepsilon$.

By our assumption, for $(x,y) \in \Omega_\varepsilon, t \in [0,T]$, we have

$$\begin{aligned} \big|f(t,x(t),y(t))\big| \quad & \leq \big|f(t,x(t),y(t)) - f(t,0,0)\big| + \big|f(t,0,0)\big|, \\ & \leq \phi\Big(\big|x(t)\big| + \big|y(t)\big|\Big) + f_0 \leq \phi(\|x\| + \|y\|) + f_0, \\ & \leq \phi\varepsilon + f_0, \end{aligned}$$

and

$$\begin{aligned} \big|g(t,x(t),y(t))\big| \quad & \leq \psi\Big(\big|x(t)\big| + \big|y(t)\big|\Big) + g_0 \leq \psi(\|x\| + \|y\|) + g_0, \\ & \leq \psi\varepsilon + g_0, \end{aligned}$$

which lead to

$$\begin{aligned}\Big|G_1(x,y)(t)\Big| \leq\ & \tfrac{T}{|\Delta|}\Big(\Big|\eta\Big|T\int_0^{\rho}\tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds(\psi(\|x\|+\|y\|)+g_0)\\ & +T\int_0^{T}\tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds(\phi(\|x\|+\|y\|)+f_0)\\ & +\Big|\eta\zeta\Big|\int_0^{\mu}\tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds(\phi(\|x\|+\|y\|)+f_0)\\ & +\Big|\eta\Big|\int_0^{T}\tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds(\psi(\|x\|+\|y\|)+g_0)\Big)\\ & +\sup_{0\leq t\leq T}\int_0^{t}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds(\phi(\|x\|+\|y\|)+f_0)\\ & \leq(\phi(\|x\|+\|y\|)+f_0)\Big[\tfrac{T}{|\Delta|}\Big(\tfrac{T^{\alpha+1}}{\Gamma(\alpha+1)}+\tfrac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\Big)+\tfrac{T^{\alpha}}{\Gamma(\alpha+1)}\Big]\\ & +(\psi(\|x\|+\|y\|)+g_0)\Big[\tfrac{T}{|\Delta|}\Big(\tfrac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)}+\tfrac{|\eta|T^{\beta}}{\Gamma(\beta+1)}\Big)\Big]\\ & \leq(\phi(\|x\|+\|y\|)+f_0)Q_1+(\psi(\|x\|+\|y\|)+g_0)Q_2\\ & \leq(\phi\varepsilon+f_0)Q_1+(\psi\varepsilon+g_0)Q_2.\end{aligned}$$

In a similar manner:

$$\Big|G_2(x,y)(t)\Big| \leq (\phi(\|x\|+\|y\|)+f_0)Q_3+(\psi(\|x\|+\|y\|)+g_0)Q_4 \ \leq(\phi\varepsilon+f_0)Q_3+(\psi\varepsilon+g_0)Q_4.$$

Hence,

$$\|G_1(x,y)\| \leq (\phi\varepsilon+f_0)Q_1+(\psi\varepsilon+g_0)Q_2,$$

and

$$\|G_2(x,y)\| \leq (\phi\varepsilon+f_0)Q_3+(\psi\varepsilon+g_0)Q_4.$$

Consequently,

$$\|G(x,y)\| \leq (\phi\varepsilon+f_0)(Q_1+Q_3)+(\psi\varepsilon+g_0)(Q_2+Q_4)\leq\varepsilon\ .$$

and we get $\|G(x,y)\|\leq\varepsilon$ that is $G\Omega_\varepsilon\subseteq\Omega_\varepsilon$.

Now let $(x_1,y_1),(x_2,y_2)\in C([0,T],\mathbb{R})\times\ C([0,T],\mathbb{R}),\forall t\in[0,T]$.

Then we have

$$\begin{aligned}\Big|G_1(x_1,y_1)(t)-\ & G_1(x_2,y_2)(t)\Big|\\ & \leq\tfrac{T}{|\Delta|}\Big(\Big|\eta\Big|T\int_0^{\rho}\tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds\psi(\|x_2-x_1\|+\|y_2-y_1\|)\\ & +T\int_0^{T}\tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\phi(\|x_2-x_1\|+\|y_2-y_1\|)\\ & +\Big|\eta\zeta\Big|\int_0^{\mu}\tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\phi(\|x_2-x_1\|+\|y_2-y_1\|)\\ & +\Big|\eta\Big|\int_0^{T}\tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\psi(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\\ & +\sup_{0\leq t\leq T}\int_0^{t}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\phi(\|x_2-x_1\|+\|y_2-y_1\|),\end{aligned}$$

$$\|G_1(x_1,y_1)-G_1(x_2,y_2)\|\leq Q_1\phi(\|x_2-x_1\|+\|y_2-y_1\|)+Q_2\psi(\|x_2-x_1\|+\|y_2-y_1\|). \tag{10}$$

and likewise

$$\|G_2(x_1,y_1)-G_2(x_2,y_2)\|\leq Q_3\phi(\|x_2-x_1\|+\|y_2-y_1\|)+Q_4\psi(\|x_2-x_1\|+\|y_2-y_1\|). \tag{11}$$

From (11) and (12) we have

$$\|G(x_1,y_1)-G(x_2,y_2)\|\leq(\phi(Q_1+Q_3)+\psi(Q_2+Q_4))(\|x_2-x_1\|+\|y_2-y_1\|).$$

Since $\phi(Q_1 + Q_3) + \psi(Q_2 + Q_4) < 1$, therefore, the operator G is a contraction operator. Hence, by Banach's fixed-point theorem, the operator G has a unique fixed point, which is the unique solution of the BVP (1) and (2). This completes the proof. □

Next we will prove the existence of solutions by applying the Leray-Schauder alternative.

Lemma 2. *"(Leray-Schauder alternative [7], p. 4) Let $F : E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let $E(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}$. Then either the set $E(F)$ is unbounded or F has at least one fixed point)".*

Theorem 2. *Assume $f, g : C([0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist $\theta_1, \theta_2, \lambda_1, \lambda_2 \geq 0$ where $\theta_1, \theta_2, \lambda_1, \lambda_2$ are real constants and $\theta_0, \lambda_0 > 0$ such that $\forall x_i, y_i \in \mathbb{R}$, $(i = 1, 2)$, we have*

$$\left|f(t, x_1, x_2)\right| \leq \theta_0 + \theta_1|x_1| + \theta_2|x_2|,$$

$$\left|g(t, x_1, x_2)\right| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|,$$

If

$$(Q_1 + Q_3)\theta_1 + (Q_2 + Q_4)\lambda_1 < 1,$$

and

$$(Q_1 + Q_3)\theta_2 + (Q_2 + Q_4)\lambda_2 < 1,$$

where $Q_i, i = 1, 2, 3, 4$ are defined in (10), then the problem (1) and (2) has at least one solution.

Proof. This proof will be presented in two steps.

Step 1: We will show that $G : C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$ is completely continuous. The continuity of the operator G holds by the continuity of the functions f, g.

Let $B \subseteq C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$ be bounded. Then there exists positive constants k_1, k_2 such that

$$\left|f(t, x(t), y(t))\right| \leq k_1, \quad \left|g(t, x(t), y(t))\right| \leq k_2, \quad \forall t \in [0, T].$$

Then $\forall (x, y) \in B$, and we have

$$\left|G_1(x, y)(t)\right| \leq Q_1 k_1 + Q_2 k_2,$$

which implies

$$\|G_1(x, y)\| \leq Q_1 k_1 + Q_2 k_2,$$

and similarly

$$\|G_2(x, y)\| \leq Q_3 k_1 + Q_4 k_2.$$

Thus, from the above inequalities, it follows that the operator G is uniformly bounded, since

$$\|G(x, y)\| \leq (Q_1 + Q_3)k_1 + (Q_2 + Q_4)k_2.$$

Next, we will show that operator G is equicontinuous. Let $\omega_1, \omega_2 \in [0, T]$ with $\omega_1 < \omega_2$. This yields

$$\begin{aligned}
\Big|G_1(x,y)(\omega_2) - \quad & G_1(x,y)(\omega_1)\Big| \\
& \leq \tfrac{\omega_2-\omega_1}{|\Delta|}\Big(\Big|\eta\Big|T\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}\Big|g(s,x(s),y(s))\Big|ds \\
& + T\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f(s,x(s),y(s))\Big|ds + \Big|\eta\zeta\Big|\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big|f(s,x(s),y(s))\Big|ds \\
& + \Big|\eta\Big|\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}\Big|g(s,x(s),y(s))\Big|ds\Big) \\
& + \Big|\int_0^{\omega_2} \tfrac{(\omega_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),y(s))ds \\
& - \int_0^{\omega_1} \tfrac{(\omega_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),y(s))ds\Big| \\
& \leq \tfrac{\omega_2-\omega_1}{|\Delta|}\Big(\Big|\eta\Big|Tk_2\int_0^\rho \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds + Tk_1\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \Big|\eta\zeta\Big|k_1\int_0^\mu \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds \\
& + \Big|\eta\Big|k_2\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\Big) + \Big|\int_0^{\omega_1}\Big(\tfrac{(\omega_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \tfrac{(\omega_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)f(s,x(s),y(s))ds\Big| \\
& + \Big|\int_{\omega_1}^{\omega_2} \tfrac{(\omega_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),y(s))ds\Big|, \\
& \leq \tfrac{\omega_2-\omega_1}{|\Delta|}\Big(\tfrac{k_2|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \tfrac{k_1T^{\alpha+1}}{\Gamma(\alpha+1)} + \tfrac{k_1|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \tfrac{k_2|\eta|T^\beta}{\Gamma(\beta+1)}\Big) \\
& + \tfrac{k_1}{\Gamma(\alpha)}\Big(\int_0^{\omega_1}\Big((\omega_2-s)^{\alpha-1} - (\omega_1-s)^{\alpha-1}\Big)ds + \int_{\omega_1}^{\omega_2}(\omega_2-s)^{\alpha-1}ds\Big).
\end{aligned}$$

And we obtain

$$\begin{aligned}
\Big|G_1(x,y)(\omega_2) - G_1(x,y)(\omega_1)\Big| \quad & \leq \tfrac{\omega_2-\omega_1}{|\Delta|}\Big(\tfrac{k_2|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \tfrac{k_1T^{\alpha+1}}{\Gamma(\alpha+1)} + \tfrac{k_1|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \tfrac{k_2|\eta|T^\beta}{\Gamma(\beta+1)}\Big) \\
& + \tfrac{k_1}{\Gamma(\alpha+1)}[\omega_2{}^\alpha - \omega_1{}^\alpha].
\end{aligned}$$

Hence, we have $\|G_1(x,y)(\omega_2) - G_1(x,y)(\omega_1)\| \to 0$ independent of x and y as $\omega_2 \to \omega_1$. Furthermore, we obtain

$$\begin{aligned}
\Big|G_2(x,y)(\omega_2) - G_2(x,y)(\omega_1)\Big| \quad & \leq \tfrac{\omega_2-\omega_1}{|\Delta|}\Big(\tfrac{k_2|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)} + \tfrac{k_1|\zeta|T^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{k_1T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \tfrac{k_2T^{\beta+1}}{\Gamma(\beta+1)}\Big) \\
& + \tfrac{k_2}{\Gamma(\beta+1)}\Big[\omega_2{}^\beta - \omega_1{}^\beta\Big],
\end{aligned}$$

which implies that $\|G_2(x,y)(\omega_2) - G_2(x,y)(\omega_1)\| \to 0$ independent of x and y as $\omega_2 \to \omega_1$.

Therefore, operator $G(x,y)$ is equicontinuous, and thus $G(x,y)$ is completely continuous.

Step 2: (Boundedness of operator)

Finally, we will show that $Z = \{(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) : (x,y) = hG(x,y), h \in [0,1]\}$ is bounded. Let $(x,y) \in \mathbb{R}$, with $(x,y) = hG(x,y)$ for any $t \in [0,T]$, we have

$$x(t) = hG_1(x,y)(t), \;\; y(t) = hG_2(x,y)(t).$$

Then

$$\Big|x(t)\Big| \leq Q_1\Big(\theta_0 + \theta_1\Big|x(t)\Big| + \theta_2\Big|y(t)\Big|\Big) + Q_2\Big(\lambda_0 + \lambda_1\Big|x(t)\Big| + \lambda_2\Big|y(t)\Big|\Big),$$

and

$$\Big|y(t)\Big| \leq Q_3\Big(\theta_0 + \theta_1\Big|x(t)\Big| + \theta_2\Big|y(t)\Big|\Big) + Q_4\Big(\lambda_0 + \lambda_1\Big|x(t)\Big| + \lambda_2\Big|y(t)\Big|\Big).$$

Hence,

$$\|x\| \leq Q_1(\theta_0 + \theta_1\|x\| + \theta_2\|y\|) + Q_2(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|),$$

and

$$\|y\| \leq Q_3(\theta_0 + \theta_1\|x\| + \theta_2\|y\|) + Q_4(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|),$$

which implies

$$\begin{aligned} \|x\| + \|y\| &\le (Q_1+Q_3)\theta_0 + (Q_2+Q_4)\lambda_0 + ((Q_1+Q_3)\theta_1 + (Q_2+Q_4)\lambda_1)\|x\| \\ &+((Q_1+Q_3)\theta_2 + (Q_2+Q_4)\lambda_2)\|y\|. \end{aligned}$$

Therefore,

$$\|(x,y)\| \le \frac{(Q_1+Q_3)\theta_0 + (Q_2+Q_4)\lambda_0}{Q_0},$$

where $Q_0 = min\{1-(Q_1+Q_3)\theta_1 - (Q_2+Q_4)\lambda_1, 1-(Q_1+Q_3)\theta_2 - (Q_2+Q_4)\lambda_2\}$. This proves that Z is bounded and hence by Leray-Schauder alternative theorem, operator G has at least one fixed point. Therefore, the BVP (1) and (2) has at least one solution on $[0,T]$. This completes the proof. □

4. Hyers-Ulam Stability

In this section, we will discuss the Hyers-Ulam stability of the solutions for the BVP (1) and (2) by means of integral representation of its solution given by

$$x(t) = G_1(x,y)(t), y(t) = G_2(x,y)(t),$$

where G_1 and G_2 are defined by (8) and (9).

Define the following nonlinear operators $N_1, N_2 \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$;

$$\begin{aligned} {}^cD^{\alpha}x(t) - f(t,x(t),y(t)) &= N_1(x,y)(t), \quad t \in [0,T], \\ {}^cD^{\beta}y(t) - g(t,x(t),y(t)) &= N_2(x,y)(t), \quad t \in [0,T]. \end{aligned}$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequality:

$$N_1(x,y) \le \varepsilon_1, \quad N_2(x,y) \le \varepsilon_2. \tag{12}$$

Definition 3. ([8,9]). *The coupled system (1) and (2) is said to be Hyers-Ulam stable, if there exist $M_1, M_2 > 0$, such that for every solution $(x^*,y^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of the inequality (13), there exists a unique solution $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of problems (1) and (2) with*

$$\|(x,y) - (x^*,y^*)\| \le M_1\varepsilon_1 + M_2\varepsilon_2.$$

Theorem 3. *Let the assumptions of Theorem 1 hold. Then the BVP (1) and (2) is Hyers-Ulam-stable.*

Proof. Let $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ be the solution of the problems (1) and (2) satisfying (8) and (9). Let (x^*,y^*) be any solution satisfying (13):

$$\begin{aligned} {}^cD^{\alpha}x^*(t) &= f(t,x^*(t),y^*(t)) + N_1(x^*,y^*)(t), \quad t \in [0,T], \\ {}^cD^{\beta}y^*(t) &= g(t,x^*(t),y^*(t)) + N_2(x^*,y^*)(t), \quad t \in [0,T]. \end{aligned}$$

So

$$\begin{aligned} x^*(t) = \; & G_1(x^*,y^*)(t) \\ & + \tfrac{t}{\Delta}\Big(\eta T \int_0^{\rho} \tfrac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} N_2(x^*,y^*)(s)ds - T\int_0^T \tfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} N_1(x^*,y^*)(s)ds \\ & + \eta\zeta \int_0^{\mu} \tfrac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} N_1(x^*,y^*)(s)ds - \eta \int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)} N_2(x^*,y^*)(s)ds\Big) \\ & + \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} N_1(x^*,y^*)(s)ds, \end{aligned}$$

It follows that

$$\begin{aligned}\Big|G_1(x^*,y^*)(t)- \quad & x^*(t)\Big| \\ & \leq \frac{T}{|\Delta|}\Big(\big|\eta\big|T\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds\varepsilon_2 + T\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\varepsilon_1 + \big|\eta\zeta\big|\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\varepsilon_1 \\ & +\big|\eta\big|\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\varepsilon_2\Big) + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\varepsilon_1, \\ & \leq \Big[\frac{T}{|\Delta|}\Big(\frac{T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\Big) + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big]\varepsilon_1 + \frac{T}{|\Delta|}\Big(\frac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \frac{|\eta|T^\beta}{\Gamma(\beta+1)}\Big)\varepsilon_2\,, \\ & \leq Q_1\varepsilon_1 + Q_2\varepsilon_2.\end{aligned}$$

Similarly,

$$\begin{aligned}\Big|G_1(x^*,y^*)(t)-x^*(t)\Big| & \leq \frac{T}{|\Delta|}\Big(\frac{|\zeta|T^\alpha}{\Gamma(\alpha+1)} + \frac{T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\Big)\varepsilon_1 + \Big[\frac{T}{|\Delta|}\Big(\frac{|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)} + \frac{T^{\beta+1}}{\Gamma(\beta+1)}\Big) + \frac{T^\beta}{\Gamma(\beta+1)}\Big], \\ & \leq Q_3\varepsilon_1 + Q_4\varepsilon_2,\end{aligned}$$

where $Q_i, i = 1,\ 2,\ 3,\ 4$ are defined in (10).

Therefore, we deduce by the fixed-point property of operator G, that is given by (8) and (9), which

$$\begin{aligned}\Big|x(t)-x^*(t)\Big| & = \Big|x(t)-G_1(x^*,y^*)(t)+G_1(x^*,y^*)(t)-x^*(t)\Big| \\ & \leq \Big|G_1(x,y)(t)-G_1(x^*,y^*)(t)\Big| + \Big|G_1(x^*,y^*)(t)-x^*(t)\Big| \\ & \leq (Q_1\phi+Q_2\psi)(x,y)-(x^*,y^*)+Q_1\varepsilon_1+Q_2\varepsilon_2,\end{aligned} \tag{13}$$

and similarly

$$\begin{aligned}\Big|y(t)-y^*(t)\Big| & = \Big|y(t)-G_2(x^*,y^*)(t)+G_2(x^*,y^*)(t)-y^*(t)\Big| \\ & \leq \Big|G_2(x,y)(t)-G_2(x^*,y^*)(t)\Big| + \Big|G_2(x^*,y^*)(t)-y^*(t)\Big| \\ & \leq (Q_3\phi+Q_4\psi)(x,y)-(x^*,y^*)+Q_3\varepsilon_1+Q_4\varepsilon_2,\end{aligned} \tag{14}$$

From (14) and (15) it follows that

$$\|(x,y)-(x^*,y^*)\| \leq (Q_1\phi+Q_2\psi+Q_3\phi+Q_4\psi)\|(x,y)-(x^*,y^*)\| + (Q_1+Q_3)\varepsilon_1 + (Q_2+Q_4)\varepsilon_2,$$

$$\begin{aligned}\|(x,y)-(x^*,y^*)\| & \leq \frac{(Q_1+Q_3)\varepsilon_1+(Q_2+Q_4)\varepsilon_2}{1-((Q_1+Q_3)\phi+(Q_2+Q_4)\psi)}, \\ & \leq M_1\varepsilon_1 + M_2\varepsilon_2.\end{aligned}$$

with

$$M_1 = \frac{(Q_1+Q_3)}{1-((Q_1+Q_3)\phi+(Q_2+Q_4)\psi)},$$
$$M_2 = \frac{(Q_2+Q_4)}{1-((Q_1+Q_3)\phi+(Q_2+Q_4)\psi)}.$$

Thus, sufficient conditions for the Hyers-Ulam stability of the solutions are obtained. □

5. Examples

Example 1. *Consider the following coupled system of fractional differential equations*

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{1}{6\pi\sqrt{81+t^2}}\Big(\frac{|x(t)|}{3+|x(t)|} + \frac{|y(t)|}{5+|x(t)|}\Big), \\ {}^cD^{\frac{7}{4}}y(t) = \frac{1}{12\pi\sqrt{64+t^2}}(sin(x(t)) + sin(y(t))), \\ x(1) = 2y'(1),\ \ y(1) = -x'(1/2),\ \ x(0) = 0,\ \ y(0) = 0, \end{cases} \tag{15}$$

$$\alpha = \frac{3}{2}, \beta = \frac{7}{4}, T = 1, \eta = 2, \zeta = -1, \mu = \frac{1}{2}, \rho = 1.$$

Using the given data, we find that $\Delta = 3, Q_1 = 1.269, Q_2 = 1.1398, Q_3 = 0.5167, Q_4 = 1.554, \phi = \frac{1}{54\pi}, \psi = \frac{1}{48\pi}$.

It is clear that

$$f(t, x(t), y(t)) = \frac{1}{6\pi\sqrt{81+t^2}}\left(\frac{|x(t)|}{3+|x(t)|} + \frac{|y(t)|}{5+|x(t)|}\right),$$

and

$$g(t, x(t), y(t)) = \frac{1}{12\pi\sqrt{64+t^2}}(sin(x(t)) + sin(y(t))),$$

are jointly continuous functions and Lipschitz function with $\phi = \frac{1}{54\pi}, \psi = \frac{1}{48\pi}$. *Moreover,*

$$\frac{1}{54\pi}(1.269 + 0.5167) + \frac{1}{48\pi}(1.1398 + 1.554) = 0.0283 < 1.$$

Thus, all the conditions of Theorem 1 are satisfied, then problem (16) has a unique solution on $[0, 1]$, *which is Hyers-Ulam-stable.*

Example 2. *Consider the following system of fractional differential equation*

$$\begin{cases} {}^cD^{5/3}x(t) = \frac{1}{80+t^4} + \frac{|x(t)|}{120(1+y^2(t))} + \frac{1}{4\sqrt{2500+t^2}}e^{-3t}\cos(y(t)),\ t \in [0,1] \\ {}^cD^{6/5}y(t) = \frac{1}{\sqrt{16+t^2}}\cos t + \frac{1}{150}e^{-3t}sin(y(t)) + \frac{1}{180}x(t),\ t \in [0,1] \\ x(1) = -3y'(1/3),\quad y(1) = x'(1),\quad x(0) = 0,\quad y(0) = 0, \end{cases} \tag{16}$$

$$\alpha = \frac{5}{3}, \beta = \frac{6}{5}, T = 1, \eta = -3, \zeta = 1, \mu = 1, \rho = 1/3.$$

Using the given data, we find that $\Delta = 3, Q_1 = 1.269, Q_2 = 1.1398, Q_3 = 0.5167, Q_4 = 1.554, \phi = \frac{1}{54\pi}, \psi = \frac{1}{48\pi}$.

It is clear that

$$|f(t,x,y)| \le \frac{1}{80} + \frac{1}{120}|x| + \frac{1}{200}|y|,$$

$$|g(t,x,y)| \le \frac{1}{4} + \frac{1}{180}|x| + \frac{1}{150}|y|.$$

Thus, $\theta_0 = \frac{1}{80}, \theta_1 = \frac{1}{120}, \theta_2 = \frac{1}{200}, \lambda_0 = \frac{1}{4}, \lambda_1 = \frac{1}{180}, \lambda_2 = \frac{1}{150}$.

Note that $(Q_1 + Q_3)\theta_1 + (Q_2 + Q_4)\lambda_1 = 0.0298 < 1$ *and* $(Q_1 + Q_3)\theta_2 + (Q_2 + Q_4)\lambda_2 = 0.0269 < 1$, *and hence by Theorem 2, problem (17) has at least one solution on* $[0, 1]$.

6. Conclusions

In this paper, the existence, uniqueness and the Hyers-Ulam stability of solutions for a coupled system of nonlinear fractional differential equations with boundary conditions were established and discussed.

Future studies may focus on different concepts of stability and existence results to a neutral time-delay system/inclusion, time-delay system/inclusion with finite delay.

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