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A Probabilistic Proof for Representations of the Riemann Zeta Function

Jiamei Liu, Yuxia Huang and Chuancun Yin *

School of Statistics, Qufu Normal University, Shandong 273165, China; 18463757687@163.com (J.L.); yxhuang1216@163.com (Y.H.)
* Correspondence: ccyin@qfnu.edu.cn

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Abstract: In this paper, we present a different proof of the well known recurrence formula for the Riemann zeta function at positive even integers, the integral representations of the Riemann zeta function at positive integers and at fractional points by means of a probabilistic approach.

Keywords: Bernoulli numbers; (half-) logistic distribution; integral representation; probabilistic approach; Riemann zeta function

1. Introduction

The well known Riemann zeta function ζ is defined by

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s}, & \text{if } \Re(s) > 1, \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, & \text{if } \Re(s) > 0, s \neq 1, \end{cases}$$

which can be continued meromorphically to the whole complex s-plane, except for a simple pole at $s = 1$, see [1–3] for details. Finding recurrence formulas and integral representations of the zeta function zeta function has become an important issue in complex analysis and number theory. One of the famous formulas is the following recursion formula for positive even integers

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}, n \in N_0, \tag{1}$$

where $N_0 = N \cup \{0\}$ and B_n is the nth Bernoulli number. Here N is the set of positive integers. Several new proofs to (1) can be found in [4–7]. A new parameterized series representation of zeta function is derived in [8]. However, no similar closed-form representation of $\zeta(s)$ at odd integers or fractional points can be found in literature. The Riemann zeta function for positive odd integer arguments can be expressed by series and integrals. One possible integral expression is established by [9] as follows

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(u) \cot(\pi u) du \ , \quad n \in N \ , \tag{2}$$

where $B_n(x)$ are Bernoulli polynomials defined by the generating function [10]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ , \quad |t| < 2\pi \ .$$

The Bernoulli numbers $B_n = B_n(0)$ are well-tabulated (see, for example, [3]):

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_{2n+1} = 0 \quad (n = 1, 2, \cdots), \cdots .$$

More lists of Bernoulli numbers and their estimation can be found in the recent work by Qi [11].

The zeta function $\zeta(s)$ has many integral representations, one of which is the following [12] (P.172) (note that there is an extra 2 in (51) of [12] (P.172):

$$\zeta(s) = \frac{\left(1 - 2^{1-s}\right)^{-1}}{\Gamma(s+1)} \int_0^\infty \frac{t^s e^t}{(e^t+1)^2} dt \ , \quad \Re(s) > 0 \ . \tag{3}$$

The aim of this note is to present a new proof of (1) for $\zeta(2n)$ and deduce the integral representations for $\zeta(n)$ and $\zeta\left(n - \frac{1}{2}\right)$. The proofs are based on the characteristic function and the moment generating function of logistic, half-logistic and elliptical symmetric logistic distributions in probability theory and mathematical statistics.

2. The Main Results and Their Proofs

In this section we present a new proof to the following results by using a probabilistic method. To the best of our knowledge, the result (5) is new.

Proposition 1. *For Riemann's zeta function ζ, we have*

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n} \ , \quad n \in N_0 \ , \tag{4}$$

$$\zeta\left(n - \frac{1}{2}\right) = \frac{2^n \int_1^\infty \frac{(\ln y)^{n-\frac{1}{2}}}{(1+y)^2} dy}{\sqrt{\pi}(2n-1)!!\left(1 - 2^{-\frac{2n-3}{2}}\right)} \ , \quad n \in N \ , \tag{5}$$

and

$$\zeta(n) = \frac{\left(1 - 2^{1-n}\right)^{-1}}{n!} \int_0^\infty \frac{x^n e^{-x}}{(1+e^{-x})^2} dx \ , \quad n \in N, n > 1 \ , \tag{6}$$

where B_n is the nth Bernoulli number.

To prove the proposition, we need the following three lemmas.

Lemma 1. *We assume that random variable X has the standard logistic distribution with the probability density function (pdf)*

$$f(x) = \frac{\exp(-x)}{(1+\exp(-x))^2} \ , \quad -\infty < x < \infty \ . \tag{7}$$

Then the moment generating function (mgf) of X is given by

$$E[\exp(tX)] = \frac{\pi t}{\sin(\pi t)} \ , \quad |t| < 1 \ . \tag{8}$$

Proof. See Johnson et al. [13] (Equation (23.10)). □

As far as we are aware, the formulas for characteristic and moment generating functions given in the following two lemmas are new.

Lemma 2. *We assume that random variable* X *has the standard* 1*-dimensional elliptically symmetric logistic distribution with pdf*

$$f(x) = c\frac{\exp(-x^2)}{(1+\exp(-x^2))^2} \ , \quad -\infty < x < \infty \ , \tag{9}$$

where

$$c = \left(\int_0^\infty t^{-\frac{1}{2}} \frac{e^{-t}}{(1+e^{-t})^2} dt \right)^{-1} .$$

Then the characteristic function of X is given by

$$E[\exp(itX)] = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c\sqrt{\pi}}{2^{2n+1}} \frac{t^{2n}}{n!} \left(1 - 2^{-\frac{2n-3}{2}}\right) \zeta\left(n - \frac{1}{2}\right), \tag{10}$$

where ζ *is the Riemann zeta function.*

Proof. Using the Taylor expansion

$$(1+\exp(-x^2))^{-2} = \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-(k-1)x^2) \ , \quad x \neq 0 \ ,$$

f in (9) can be written as

$$f(x) = c\sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-kx^2) \ , \quad x \neq 0 \ .$$

Noting that $f(-x) = f(x), \quad -\infty < x < \infty$, we only need to determine the even-order moments. For $m \geq 1$ we get

$$\begin{aligned} E(X^{2m}) &= 2\int_0^\infty x^{2m} f(x)dx = 2c\int_0^\infty x^{2m} \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-kx^2)dx \\ &= c\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sqrt{\pi}}{2^{2m}} \frac{(2m)!}{m!} k^{-\frac{2m-1}{2}} \\ &= \frac{\sqrt{\pi}c}{2^{2m}} \frac{(2m)!}{m!} \left(1 - 2^{-\frac{2m-3}{2}}\right) \zeta\left(m - \frac{1}{2}\right), \end{aligned}$$

where we have used the fact that

$$\int_0^\infty \exp(-bx^2)x^{2k}dx = \frac{\sqrt{\pi}}{2} \frac{1}{2} \frac{3}{2} \cdots \frac{2k-1}{2} b^{-\frac{2k+1}{2}} .$$

For any $t \in (-\infty, \infty)$, we get the characteristic function of X by performing the following calculations

$$\begin{aligned} E[\exp(itX)] &= E\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}X^{2n}}{(2n)!}\right] = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}E(X^{2n})}{(2n)!} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c\sqrt{\pi}}{2^{2n}} \frac{t^{2n}}{n!} \left(1 - 2^{-\frac{2n-3}{2}}\right) \zeta\left(n - \frac{1}{2}\right). \end{aligned}$$

This ends the proof of Lemma 2. □

Lemma 3. *We assume that random variable X has the standard half-logistic distribution with the pdf*

$$f(x) = \frac{2\exp(-x)}{(1+\exp(-x))^2}, \quad x > 0. \tag{11}$$

Then the mgf of X is given by

$$E[\exp(tX)] = 1 + 2\sum_{n=1}^{\infty}(1-2^{1-n})\zeta(n)t^n, \quad |t| < 1, \tag{12}$$

where ζ is the Riemann zeta function.

Proof. The mean of X is given by

$$E(X) = 2\int_0^{\infty} \frac{xe^{-x}}{(1+\exp(-x))^2}dx = 2\ln 2.$$

Using the expansion

$$\begin{aligned} f(x) &= 2\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-kx} \\ &= 2\sum_{k=1}^{\infty}(2k-1)e^{-(2k-1)x} - 2\sum_{k=1}^{\infty}2ke^{-2kx}, \quad x > 0, \end{aligned}$$

we get, for any positive integer $n > 1$,

$$\begin{aligned} E(X^n) &= 2\int_0^{\infty} x^n f(x)dx \\ &= 2\sum_{k=1}^{\infty}(2k-1)\int_0^{\infty} x^n e^{-(2k-1)x}dx - 2\sum_{k=1}^{\infty}2k\int_0^{\infty} x^n e^{-2kx}dx \\ &= 2n!\sum_{k=1}^{\infty}\frac{1}{(2k-1)^n} - 2n!\sum_{k=1}^{\infty}\frac{1}{(2k)^n} \\ &= 2n!(1-2^{1-n})\zeta(n). \end{aligned}$$

Then we have

$$\begin{aligned} E(e^{tX}) &= 1 + \sum_{k=1}^{\infty}\frac{E(X^k)}{k!}t^k = 1 + 2t\ln 2 + \sum_{k=2}^{\infty}\frac{2k!\zeta(k)(1-2^{1-k})}{k!}t^k \\ &= 1 + 2t\ln 2 + 2\sum_{k=2}^{\infty}(1-2^{1-k})\zeta(k)t^k, \quad |t| < 1, \end{aligned}$$

where we have used the fact

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

This completes the proof of Lemma 3. □

Proof of Proposition 1. The mgf of the standard logistic distribution can be written as

$$E(e^{tX}) = 1 + \sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)\zeta(2n)}{2^{2(n-1)}}t^{2n}, \tag{13}$$

see, for example, [14]. Comparing (7) and (10) yields $g(t) = h(t), \quad |t| < 1$, where

$$g(t) = 1 + \sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)\zeta(2n)}{2^{2(n-1)}}t^{2n},$$

and

$$h(t) = \frac{\pi t}{\sin(\pi t)}.$$

Using the series expansion (see e.g., [15])

$$\frac{\pi t}{\sin(\pi t)} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{2^{2k}-2}{(2k)!} B_{2k} (\pi t)^{2k}, \quad |t| < 1,$$

where B_{2k} is the $2k$th Bernoulli numbers, we have

$$\sum_{n=1}^{\infty} \frac{(2^{2n-1}-1)\zeta(2n)}{2^{2(n-1)}} t^{2n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}-2}{(2n)!} B_{2n} (\pi t)^{2n}, \quad |t| < 1,$$

from which we deduce that

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}.$$

This completes the proof of (4).

Now we prove (5). Denoted by

$$H(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx,$$

and

$$G(t) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{c\sqrt{\pi}}{2^{2k+1}} \frac{t^{2k}}{k!} \Big(1 - 2^{-\frac{2k-3}{2}}\Big) \zeta\Big(k - \frac{1}{2}\Big),$$

where f is defined by (7). Taking $2n$th and $(2n+1)$th derivatives of the two functions with respect to t, we get

$$H^{(2n)}(t) = (-1)^n 2c \int_0^{\infty} x^{2n} \cos tx \frac{\exp(-x^2)}{(1+\exp(-x^2))^2} dx,$$

$$H^{(2n+1)}(t) = (-1)^{n+1} 2c \int_0^{\infty} x^{2n-1} \sin tx \frac{\exp(-x^2)}{(1+\exp(-x^2))^2} dx,$$

and

$$G^{(2n-1)}(t) = \sum_{k=n}^{\infty} (-1)^k \frac{c\sqrt{\pi} \prod_{l=0}^{2n-2} (2k-l)}{2^{2k} k!} \Big(1 - 2^{-\frac{2k-3}{2}}\Big) \zeta\Big(k - \frac{1}{2}\Big) t^{2k-2n+1},$$

$$G^{(2n)}(t) = \sum_{k=n}^{\infty} (-1)^k \frac{c\sqrt{\pi} \prod_{l=0}^{2n-1} (2k-l)}{2^{2k} k!} \Big(1 - 2^{-\frac{2k-3}{2}}\Big) \zeta\Big(k - \frac{1}{2}\Big) t^{2k-2n}.$$

Note that $H(t) = G(t)$ for any real t, and thus $H^{(n)}(t) = G^{(n)}(t)$ for any real t and any positive integers n. In particular, $H^{(n)}(0) = G^{(n)}(0)$. However, $H^{(2n+1)}(0) = G^{(2n+1)}(0) = 0$, and from $H^{(2n)}(0) = G^{(2n)}(0)$ we have

$$\zeta\left(\frac{2n-1}{2}\right) = \frac{2^{2n+1}n!\int_0^\infty x^{2n}\frac{\exp(-x^2)}{(1+\exp(-x^2))^2}dx}{\sqrt{\pi}(2n)!\left(1-2^{-\frac{2n-3}{2}}\right)}$$
$$= \frac{2^n\int_0^\infty x^{\frac{2n-1}{2}}\frac{\exp(-x)}{(1+\exp(-x))^2}dx}{\sqrt{\pi}(2n-1)!!\left(1-2^{-\frac{2n-3}{2}}\right)}$$
$$= \frac{2^n\int_1^\infty \frac{(\ln y)^{n-\frac{1}{2}}}{(1+y)^2}dy}{\sqrt{\pi}(2n-1)!!\left(1-2^{-\frac{2n-3}{2}}\right)}, \quad n \in N,$$

which concludes the proof of (5).

Finally, we prove (6). Using (12) one has

$$2\int_0^\infty e^{tx}\frac{e^{-x}}{(1+e^{-x})^2}dx = 1+2t\ln 2+2\sum_{k=2}^\infty\left(1-2^{1-k}\right)\zeta(k)t^k, \quad |t|<1. \tag{14}$$

Taking the nth derivative of both sides of (14) with respect to t and then setting $t=0$ yields the desired result. □

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