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p -Regularity and p -Regular Modification in \top -Convergence Spaces

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Abstract: Fuzzy convergence spaces are extensions of convergence spaces. \top -convergence spaces are important fuzzy convergence spaces. In this paper, p -regularity (a relative regularity) in \top -convergence spaces is discussed by two equivalent approaches. In addition, lower and upper p -regular modifications in \top -convergence spaces are further investigated and studied. Particularly, it is shown that lower (resp., upper) p -regular modification and final (resp., initial) structures have good compatibility.

Keywords: fuzzy topology; fuzzy convergence; \top -convergence space; regularity

1. Introduction

Convergence spaces [1] are generalizations of topological spaces. Regularity is an important property in convergence spaces. In general, there are two equivalent approaches to characterize regularity. One approach is stated through a diagonal condition of filters [2,3], the other approach is represented through a closure condition of filters [4]. In [5,6], for a pair of convergence structures p, q on the same underlying set, Wilde-Kent-Richardson considered a relative regularity (called p -regularity) both from two equivalent approaches. When $p = q$, p -regularity is nothing but regularity. Wilde-Kent [6] further presented a theory of lower and upper p -regular modifications in convergence spaces. Said precisely, for convergence structures p, q on a set X , the lower (resp., upper) p -regular modification of q is defined as the finest (resp., coarsest) p -regular convergence structure coarser (resp., finer) than q .

Fuzzy convergence spaces are natural extensions of convergence spaces. Quite recently, two types of fuzzy convergence spaces received wide attention: (1) stratified L -generalized convergence spaces (resp., stratified L -convergence spaces) initiated by Jäger [7] (resp., Flores [8]) and then developed by many scholars [8–30]; and (2) \top -convergence spaces introduced by Fang [31] and then discussed by many researchers [32–36]. Regularity in stratified L -generalized convergence spaces (resp., stratified L -convergence spaces) was studied by Jäger [37] (resp., Boustique-Richardson [38,39]), p -regularity and p -regular modifications in stratified L -generalized convergence spaces and that in stratified L -convergence spaces were discussed by Li [40,41]. Regularity in \top -convergence spaces by different diagonal conditions of \top -filters were researched by Fang [31] and Li [42], respectively. Regularity in \top -convergence spaces by closure condition of \top -filters were studied by Reid and Richardson [36]. In this paper, we shall discuss p -regularity and p -regular modifications in \top -convergence spaces.

The contents are arranged as follows. Section 2 recalls some notions and notations for later use. Section 3 presents p -regularity in \top -convergence spaces by a diagonal condition of \top -filters and a

closure condition of \top -filters, respectively. Section 4 mainly discusses p -regular modifications in \top -convergence spaces. The lower and upper p -regular modifications in \top -convergence spaces are investigated and researched. Especially, it is shown that lower (resp., upper) p -regular modification and final (resp., initial) structures have good compatibility.

2. Preliminaries

In this paper, if not otherwise stated, $L = (L, \leq)$ is always a complete lattice with a top element \top and a bottom element \perp , which satisfies the distributive law $\alpha \wedge (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} (\alpha \wedge \beta_i)$. A lattice with these conditions is called a complete Heyting algebra. The operation $\rightarrow: L \times L \rightarrow L$ given by

$$\alpha \rightarrow \beta = \bigvee \{ \gamma \in L : \alpha \wedge \gamma \leq \beta \}.$$

is called the residuation with respect to \wedge . We collect here some basic properties of the binary operations \wedge and \rightarrow [43].

- (1) $a \rightarrow b = \top \Leftrightarrow a \leq b$;
- (2) $a \wedge b \leq c \Leftrightarrow b \leq a \rightarrow c$;
- (3) $a \wedge (a \rightarrow b) \leq b$;
- (4) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$;
- (5) $(\bigvee_{j \in J} a_j) \rightarrow b = \bigwedge_{j \in J} (a_j \rightarrow b)$; (6) $a \rightarrow (\bigwedge_{j \in J} b_j) = \bigwedge_{j \in J} (a \rightarrow b_j)$.

A function $\mu : X \rightarrow L$ is said to be an L -fuzzy set in X , and all L -fuzzy sets in X are denoted as L^X . The operators $\vee, \wedge, \rightarrow$ on L can be translated onto L^X pointwisely. Precisely, for any $\mu, \nu, \mu_t (t \in T) \in L^X$,

$$\left(\bigvee_{t \in T} \mu_t\right)(x) = \bigvee_{t \in T} \mu_t(x), \quad \left(\bigwedge_{t \in T} \mu_t\right)(x) = \bigwedge_{t \in T} \mu_t(x), \quad (\mu \rightarrow \nu)(x) = \mu(x) \rightarrow \nu(x).$$

Let $f : X \rightarrow Y$ be a function. We define $f^\rightarrow : L^X \rightarrow L^Y$ by $f^\rightarrow(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ for $\mu \in L^X$ and $y \in Y$, and define $f^\leftarrow : L^Y \rightarrow L^X$ by $f^\leftarrow(\nu)(x) = \nu(f(x))$ for $\nu \in L^Y$ and $x \in X$ [43].

Let μ, ν be L -fuzzy sets in X . The subethood degree of μ, ν , denoted as $S_X(\mu, \nu)$, is defined by $S_X(\mu, \nu) = \bigwedge_{x \in X} (\mu(x) \rightarrow \nu(x))$ [44–46]

Lemma 1. [31,42,47] Let $f : X \rightarrow Y$ be a function and $\mu_1, \mu_2 \in L^X, \lambda_1, \lambda_2 \in L^Y$. Then

- (1) $S_X(\mu_1, \mu_2) \leq S_Y(f^\rightarrow(\mu_1), f^\rightarrow(\mu_2))$,
- (2) $S_Y(\lambda_1, \lambda_2) \leq S_X(f^\leftarrow(\lambda_1), f^\leftarrow(\lambda_2))$,
- (3) $S_Y(f^\rightarrow(\mu_1), \lambda_1) = S_X(\mu_1, f^\leftarrow(\lambda_1))$.

2.1. \top -Filters and \top -Convergence Spaces

Definition 1. [43,48] A nonempty subset $\mathbb{F} \subseteq L^X$ is said to be a \top -filter on the set X if it satisfies the following three conditions:

- (TF1) $\forall \lambda \in \mathbb{F}, \bigvee_{x \in X} \lambda(x) = \top$,
- (TF2) $\forall \lambda, \mu \in \mathbb{F}, \lambda \wedge \mu \in \mathbb{F}$,
- (TF3) if $\bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda) = \top$, then $\lambda \in \mathbb{F}$.

The set of all \top -filters on X is denoted by $\mathbb{F}_L^\top(X)$.

Definition 2. [43] A nonempty subset $\mathbb{B} \subseteq L^X$ is referred to be a \top -filter base on the set X if it holds that:

- (TB1) $\forall \lambda \in \mathbb{B}, \bigvee_{x \in X} \lambda(x) = \top$,
- (TB2) if $\lambda, \mu \in \mathbb{B}$, then $\bigvee_{v \in \mathbb{B}} S_X(v, \lambda \wedge \mu) = \top$.

Each \top -filter base generates a \top -filter $\mathbb{F}_{\mathbb{B}}$ by

$$\mathbb{F}_{\mathbb{B}} := \{ \lambda \in L^X \mid \bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \top \}.$$

Example 1. [31,43] Let $f : X \rightarrow Y$ be a function.

- (1) For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the family $\{f^\rightarrow(\lambda) \mid \lambda \in \mathbb{F}\}$ forms a \top -filter base on Y . The generated \top -filter is denoted as $f^\rightarrow(\mathbb{F})$, called the image of \mathbb{F} under f . It is known that $\mu \in f^\rightarrow(\mathbb{F}) \iff f^\leftarrow(\mu) \in \mathbb{F}$.
- (2) For any $\mathbb{G} \in \mathbb{F}_L^\top(Y)$, the family $\{f^\leftarrow(\mu) \mid \mu \in \mathbb{G}\}$ forms a \top -filter base on X iff $\bigvee_{y \in f(X)} \mu(y) = \top$ holds for all $\mu \in \mathbb{G}$. The generated \top -filter (if exists) is denoted as $f^\leftarrow(\mathbb{G})$, called the inverse image of \mathbb{G} under f . It is known that $\mathbb{G} \subseteq f^\rightarrow(f^\leftarrow(\mathbb{G}))$ holds whenever $f^\leftarrow(\mathbb{G})$ exists. Furthermore, $f^\leftarrow(\mathbb{G})$ always exists and $\mathbb{G} = f^\rightarrow(f^\leftarrow(\mathbb{G}))$ whenever f is surjective.
- (3) For any $x \in X$, the family $[x]_\top =: \{ \lambda \in L^X \mid \lambda(x) = \top \}$ is a \top -filter on X , and $f^\rightarrow([x]_\top) = [f(x)]_\top$.

Lemma 2. Let $f : X \rightarrow Y$ be a function.

- (1) If \mathbb{B} is a \top -filter base of $\mathbb{F} \in \mathbb{F}_L^\top(X)$, then $\{f^\rightarrow(\lambda) \mid \lambda \in \mathbb{B}\}$ is a \top -filter base of $f^\rightarrow(\mathbb{F})$, see Example 2.9 (1) in [31].
- (2) If \mathbb{B} is a \top -filter base of $\mathbb{G} \in \mathbb{F}_L^\top(Y)$ and $f^\leftarrow(\mathbb{G})$ exists, then $\{f^\leftarrow(\mu) \mid \mu \in \mathbb{B}\}$ is a \top -filter base of $f^\leftarrow(\mathbb{G})$, see Example 2.9 (2) in [31].
- (3) Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^\top(X)$ and \mathbb{B} be a \top -filter base of \mathbb{F} . Then $\mathbb{B} \subseteq \mathbb{G}$ implies that $\mathbb{F} \subseteq \mathbb{G}$, see Lemma 2.5 (1) in [42].
- (4) Let $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and \mathbb{B} be a \top -filter base of \mathbb{F} . Then $\bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda)$, see Lemma 3.1 in [36].

For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$, we define $\Lambda(\mathbb{F}) : L^X \rightarrow L$ as

$$\forall \lambda \in L^X, \Lambda(\mathbb{F})(\lambda) = \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda),$$

then $\Lambda(\mathbb{F})$ is a tightly stratified L -filter on X [47].

In the following, we recall some notions and notations collected in [29].

Definition 3. [31] Let X be a nonempty set. Then a function $q : \mathbb{F}_L^\top(X) \rightarrow 2^X$ is said to be a \top -convergence structure on X if it satisfies the following two conditions:

- (TC1) $\forall x \in X, [x]_\top \xrightarrow{q} x$;
- (TC1) if $\mathbb{F} \xrightarrow{q} x$ and $\mathbb{F} \subseteq \mathbb{G}$, then $\mathbb{G} \xrightarrow{q} x$.

where $\mathbb{F} \xrightarrow{q} x$ is short for $x \in q(\mathbb{F})$. The pair (X, q) is said to be a \top -convergence space.

A function $f : X \rightarrow X'$ between \top -convergence spaces $(X, q), (X', q')$ is said to be continuous if $f^\rightarrow(\mathbb{F}) \xrightarrow{q'} f(x)$ for any $\mathbb{F} \xrightarrow{q} x$.

We denote the category consisting of \top -convergence spaces and continuous functions as \top -CS. It has been known that \top -CS is topological over SET [31].

For a source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$, the initial structure, q on X is defined by

$$\mathbb{F} \xrightarrow{q} x \iff \forall i \in I, f_i^\rightarrow(\mathbb{F}) \xrightarrow{q_i} f_i(x) \text{ [35,49]}.$$

For a sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$, the final structure, q on X is defined as

$$\mathbb{F} \xrightarrow{q} x \iff \begin{cases} \mathbb{F} \supseteq [x]_\top, & x \notin \bigcup_{i \in I} f_i(X_i); \\ \mathbb{F} \supseteq f_i^\rightarrow(\mathbb{G}_i), & \exists i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^\top(X_i) \text{ s.t } f(x_i) = x, \mathbb{G}_i \xrightarrow{q_i} x_i. \end{cases}$$

When $X = \cup_{i \in I} f_i(X_i)$, the final structure q can be characterized as

$$\mathbb{F} \xrightarrow{q} x \iff \mathbb{F} \supseteq f_i^{\Rightarrow}(\mathbb{G}_i) \text{ for some } \mathbb{G}_i \xrightarrow{q_i} x_i \text{ with } f(x_i) = x.$$

Let $\top(X)$ denote the set of all \top -convergence structures on a set X . For $p, q \in \top(X)$, we say that q is finer than p (or p is coarser than q), denoted as $p \leq q$ for short, if the identity $\text{id}_X : (X, q) \rightarrow (X, p)$ is continuous. It has been known that $(\top(X), \leq)$ forms a completed lattice. The discrete (resp., indiscrete) structure δ (resp., ι) is the top (resp., bottom) element of $(\top(X), \leq)$, where δ is defined as $\mathbb{F} \xrightarrow{\delta} x$ iff $\mathbb{F} \supseteq [x]_{\top}$; and ι is defined as $\mathbb{F} \xrightarrow{\iota} x$ for all $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, $x \in X$ [42].

Remark 1. When $L = \{\perp, \top\}$, \top -convergence spaces degenerate into convergence spaces. Therefore, \top -convergence spaces are natural generalizations of convergence spaces.

3. p -Regularity in \top -Convergence Spaces

In this section, we shall discuss the p -regularity in \top -convergence spaces. Two equivalent approaches are considered, one approach using diagonal \top -filter and the other approach using closure of \top -filter. Moreover, it will be proved that p -regularity is preserved under the initial and final structures in the category \top -CS.

At first, we recall the notions of diagonal \top -filter and closure of \top -filter to define p -regularity.

Let J, X be any sets and $\phi : J \rightarrow \mathbb{F}_L^{\top}(X)$ be any function. Then we define a function $\hat{\phi} : L^X \rightarrow L^J$ as

$$\forall \lambda \in L^X, \forall j \in J, \hat{\phi}(\lambda)(j) = \Lambda(\phi(j))(\lambda) = \bigvee_{\mu \in \phi(j)} S_X(\mu, \lambda).$$

For any $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$, it is known that the subset of L^X defined by

$$k\phi\mathbb{F} := \{\lambda \in L^X \mid \hat{\phi}(\lambda) \in \mathbb{F}\}$$

forms a \top -filter on X , called diagonal \top -filter of \mathbb{F} under ϕ [31]. It was shown that $S_X(\lambda, \mu) \leq S_J(\hat{\phi}(\lambda), \hat{\phi}(\mu))$ for any $\lambda, \mu \in L^X$.

Lemma 3. Let $f : X \rightarrow Y$ and $\phi : J \rightarrow \mathbb{F}_L^{\top}(X)$ be functions. Then for any $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$ we have $f^{\Rightarrow}(k\phi\mathbb{F}) = k(f^{\Rightarrow} \circ \phi)\mathbb{F}$.

Proof. $f^{\Rightarrow}(k\phi\mathbb{F}) \subseteq k(f^{\Rightarrow} \circ \phi)\mathbb{F}$. By Lemma 2 (3) we need only check that $f^{\rightarrow}(\lambda) \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$ for any $\lambda \in k\phi\mathbb{F}$. Take $\lambda \in k\phi\mathbb{F}$ then $\hat{\phi}(\lambda) \in \mathbb{F}$. Please note that $\forall j \in J$,

$$\hat{\phi}(\lambda)(j) = \bigvee_{\mu \in \phi(j)} S_X(\mu, \lambda) \leq \bigvee_{\mu \in \phi(j)} S_Y(f^{\rightarrow}(\mu), f^{\rightarrow}(\lambda)) \leq \bigvee_{v \in f^{\Rightarrow}(\phi(j))} S_Y(v, f^{\rightarrow}(\lambda)) = \widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda))(j),$$

i.e., $\hat{\phi}(\lambda) \leq \widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda))$, and so $\widehat{f^{\Rightarrow} \circ \phi}(f^{\rightarrow}(\lambda)) \in \mathbb{F}$, i.e., $f^{\rightarrow}(\lambda) \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$.

$k(f^{\Rightarrow} \circ \phi)\mathbb{F} \subseteq f^{\Rightarrow}(k\phi\mathbb{F})$. For any $\lambda \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}$ we have $\widehat{f^{\Rightarrow} \circ \phi}(\lambda) \in \mathbb{F}$. By Lemma 2 (4), $\forall j \in J$,

$$\widehat{f^{\Rightarrow} \circ \phi}(\lambda)(j) = \bigvee_{v \in f^{\Rightarrow}(\phi(j))} S_Y(v, \lambda) = \bigvee_{\mu \in \phi(j)} S_Y(f^{\rightarrow}(\mu), \lambda) \leq \bigvee_{\mu \in \phi(j)} S_X(\mu, f^{\leftarrow}(\lambda)) = \hat{\phi}(f^{\leftarrow}(\lambda))(j),$$

i.e., $\widehat{f^{\Rightarrow} \circ \phi}(\lambda) \leq \hat{\phi}(f^{\leftarrow}(\lambda))$, and so $\hat{\phi}(f^{\leftarrow}(\lambda)) \in \mathbb{F}$, i.e., $f^{\leftarrow}(\lambda) \in k\phi\mathbb{F}$ then $\lambda \in f^{\Rightarrow}(k\phi\mathbb{F})$. \square

Definition 4. [36] Let (X, p) be a \top -convergence space. For each $\lambda \in L^X$, the L -set $\bar{\lambda}_p \in L^X$ defined by

$$\forall x \in X, \bar{\lambda}_p(x) = \bigvee_{\mathbb{F} \xrightarrow{p} x} \Lambda(\mathbb{F})(\lambda) = \bigvee_{\mathbb{F} \xrightarrow{p} x} \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda)$$

is called closure of λ w.r.t p .

For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the closure of \mathbb{F} regarding p , denoted as $cl_p(\mathbb{F})$, is defined to be the \top -filter generated by $\{\bar{\lambda}_p | \lambda \in \mathbb{F}\}$ as a \top -filter base.

Lemma 4. [36] Let (X, p) be a \top -convergence space. Then for all $\lambda, \mu \in L^X$ we get

- (1) $\lambda \leq \bar{\lambda}_p$;
- (2) $\lambda \leq \mu$ implies $\bar{\lambda}_p \leq \bar{\mu}_p$;
- (3) $S_X(\lambda, \mu) \leq S_X(\bar{\lambda}, \bar{\mu})$.

Let \mathbb{N} be the set of natural numbers containing 0 and let (X, p) be a \top -convergence space. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, we define $cl_p^0(\mathbb{F}) = \mathbb{F}$. Furthermore, for any $n \in \mathbb{N}$, we define the $n + 1$ th iteration of the closure \top -filter of \mathbb{F} as $cl_p^{n+1}(\mathbb{F}) = cl_p(cl_p^n(\mathbb{F}))$ if $cl_p^n(\mathbb{F})$ has been defined.

The next proposition collects the properties of closure of \top -filters. We omit the obvious proofs.

Proposition 1. Let (X, p) be a \top -convergence space and $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^\top(X)$. Then for any $n \in \mathbb{N}$,

- (1) $cl_p^n(\mathbb{F}) \subseteq \mathbb{F}$,
- (2) if $\mathbb{F} \subseteq \mathbb{G}$, then $cl_p^n(\mathbb{F}) \subseteq cl_p^n(\mathbb{G})$,
- (3) if $p' \in T(X)$ and $p \leq p'$, then $cl_p^n(\mathbb{F}) \subseteq cl_{p'}^n(\mathbb{F})$.

Definition 5. A function $f : (X, q) \rightarrow (Y, p)$ between \top -convergence spaces is said to be a closure function if $\overline{f^\rightarrow(\lambda)}_p \leq f^\rightarrow(\bar{\lambda}_q)$ for any $\lambda \in L^X$.

Proposition 2. Suppose that $f : (X, q) \rightarrow (Y, p)$ is a function between \top -convergence spaces and $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $n \in \mathbb{N}$.

- (1) If f is a continuous function, then $f^\rightarrow(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^\rightarrow(\mathbb{F}))$.
- (2) If f is a closure function, then $f^\rightarrow(cl_q^n(\mathbb{F})) \subseteq cl_p^n(f^\rightarrow(\mathbb{F}))$.

Proof. (1) Let's prove it by mathematical induction.

Firstly, we check $\overline{f^\leftarrow(\lambda)}_q \leq f^\leftarrow(\bar{\lambda}_p)$ for any $\lambda \in L^Y$. In fact, for any $x \in X$, by continuity of f we obtain

$$\begin{aligned} \overline{f^\leftarrow(\lambda)}_q(x) &= \bigvee_{\mathbb{G} \xrightarrow{q} x} \bigvee_{\mu \in \mathbb{G}} S_X(\mu, f^\leftarrow(\lambda)) = \bigvee_{\mathbb{G} \xrightarrow{q} x} \bigvee_{\mu \in \mathbb{G}} S_Y(f^\rightarrow(\mu), \lambda) \\ &\leq \bigvee_{f^\rightarrow(\mathbb{G}) \xrightarrow{p} f(x)} \bigvee_{f^\rightarrow(\mu) \in f^\rightarrow(\mathbb{G})} S_Y(f^\rightarrow(\mu), \lambda) \leq \bigvee_{\mathbb{H} \xrightarrow{p} f(x)} \bigvee_{v \in \mathbb{H}} S_Y(v, \lambda) = f^\leftarrow(\bar{\lambda}_p)(x). \end{aligned}$$

Secondly, we prove $f^\rightarrow(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^\rightarrow(\mathbb{F}))$ when $n = 1$. Let $\lambda \in f^\rightarrow(\mathbb{F})$, i.e., $f^\leftarrow(\lambda) \in \mathbb{F}$. Then by $\overline{f^\leftarrow(\lambda)}_q \leq f^\leftarrow(\bar{\lambda}_p)$ we have $f^\leftarrow(\bar{\lambda}_p) \in cl_q(\mathbb{F})$, i.e., $\bar{\lambda}_p \in f^\rightarrow(cl_q(\mathbb{F}))$. It follows by Lemma 2 (3) that $f^\rightarrow(cl_q(\mathbb{F})) \supseteq cl_p(f^\rightarrow(\mathbb{F}))$.

Thirdly, we assume that $f^\rightarrow(cl_q^n(\mathbb{F})) \supseteq cl_p^n(f^\rightarrow(\mathbb{F}))$ when $n = k$. Then we prove $f^\rightarrow(cl_q^{k+1}(\mathbb{F})) \supseteq cl_p^{k+1}(f^\rightarrow(\mathbb{F}))$ when $n = k + 1$. In fact,

$$f^\rightarrow(cl_q^{k+1}(\mathbb{F})) = f^\rightarrow(cl_q(cl_q^k(\mathbb{F}))) \supseteq cl_p(f^\rightarrow(cl_q^k(\mathbb{F}))) \supseteq cl_p(cl_p^k(f^\rightarrow(\mathbb{F}))) = cl_p^{k+1}(f^\rightarrow(\mathbb{F})).$$

(2) We prove only that the inequality holds for $n = 1$, and the rest of the proof is similar to (1).

For any $\lambda \in \mathbb{F}$, we have $\bar{\lambda}_q \in cl_q(\mathbb{F})$ and then $f^\rightarrow(\lambda) \in f^\rightarrow(\mathbb{F})$. From f is a closure function, we conclude that $f^\rightarrow(\bar{\lambda}_q) \geq \overline{f^\rightarrow(\lambda)}_p \in cl_p(f^\rightarrow(\mathbb{F}))$ and so $f^\rightarrow(\bar{\lambda}_q) \in cl_p(f^\rightarrow(\mathbb{F}))$. By Lemma 2 (1), (3) we obtain $f^\rightarrow(cl_q(\mathbb{F})) \subseteq cl_p(f^\rightarrow(\mathbb{F}))$. \square

Now, we tend our attention to p -regularity and its equivalent characterization. In the following, we shorten a pair of \top -convergence spaces (X, p) and (X, q) as (X, p, q) .

Definition 6. Let (X, p, q) be a pair of \top -convergence spaces. Then q is said to be p -regular if the following condition p -(**TC**) is fulfilled.

$$p\text{-(TC): } \forall \mathbb{F} \in \mathbb{F}_L^\top(X), \forall x \in X, \mathbb{F} \xrightarrow{q} x \implies cl_p(\mathbb{F}) \xrightarrow{q} x.$$

Remark 2. When $L = \{\perp, \top\}$, a \top -convergence space degenerates into a convergence space, and the condition p -(**TC**) degenerates into the crisp p -regularity condition in [5]. When $p = q$, the condition p -(**TC**) is precisely the regular characterization in [36].

We say a pair of \top -convergence spaces (X, p, q) fulfill the Fischer \top -diagonal condition whenever

p -(TR): Let J, X be any sets, $\psi : J \rightarrow X$, and $\phi : J \rightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{p} \psi(j)$, for each $j \in J$. Then for each $\mathbb{F} \in \mathbb{F}_L^\top(J)$ and each $x \in X, k\phi\mathbb{F} \xrightarrow{q} x$ implies $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$.

Remark 3. When $L = \{\perp, \top\}$, a \top -convergence space degenerates into a convergence space, and the condition p -(**TC**) degenerates into the Fischer diagonal condition $R_{p,q}$ in [6]. When $p = q$, the condition p -(**TR**) is precisely the diagonal condition (**TR**) in [31].

In the following, we shall show that p -regularity can be described by Fischer \top -diagonal condition p -(**TR**).

Lemma 5. Let (X, p, q) be a pair of \top -convergence spaces and let J, X, ϕ, ψ be defined as in p -(**TR**). Then $S_X(\bar{\mu}_p, \lambda) \leq S_J(\hat{\phi}(\mu), \psi^\leftarrow(\lambda))$ for all $\lambda, \mu \in L^X$.

Proof. Let $\lambda, \mu \in L^X$.

$$\begin{aligned} S_X(\bar{\mu}_p, \lambda) &= \bigwedge_{x \in X} ([\bigvee_{\mathbb{G} \xrightarrow{p} x} \Lambda(\mathbb{G})(\mu)] \rightarrow \lambda(x)), \text{ by } \psi(j) \in X, \phi(j) \xrightarrow{p} \psi(j) \\ &\leq \bigwedge_{j \in J} (\Lambda(\phi(j))(\mu) \rightarrow \lambda(\psi(j))) = \bigwedge_{j \in J} (\hat{\phi}(\mu)(j) \rightarrow \psi^\leftarrow(\lambda)(j)) \\ &= S_J(\hat{\phi}(\mu), \psi^\leftarrow(\lambda)). \quad \square \end{aligned}$$

Theorem 1. (Theorem 4.8 in [36] for $p = q$) Let (X, p, q) be a pair of \top -convergence spaces. Then p -(**TC**) \iff p -(**TR**).

Proof. p -(**TC**) \implies p -(**TR**). Let J, X, ϕ, ψ be defined as in p -(**TR**). Assume that $\mathbb{F} \in \mathbb{F}_L^\top(J)$ and $k\phi\mathbb{F} \xrightarrow{q} x$. Then it follows by p -(**TC**) that $cl_p(k\phi\mathbb{F}) \xrightarrow{q} x$.

Next we prove that $cl_p(k\phi\mathbb{F}) \subseteq \psi^\Rightarrow(\mathbb{F})$. Indeed, for any $\lambda \in cl_p(k\phi\mathbb{F})$, we have

$$\top = \bigvee_{\mu \in k\phi\mathbb{F}} S_X(\bar{\mu}_p, \lambda) \stackrel{\text{Lemma 5}}{\leq} \bigvee_{\mu \in k\phi\mathbb{F}} S_J(\hat{\phi}(\mu), \psi^\leftarrow(\lambda)) = \bigvee_{\hat{\phi}(\mu) \in \mathbb{F}} S_J(\hat{\phi}(\mu), \psi^\leftarrow(\lambda)) \leq \bigvee_{v \in \mathbb{F}} S_J(v, \psi^\leftarrow(\lambda)),$$

which means $\psi^\leftarrow(\lambda) \in \mathbb{F}$, i.e., $\lambda \in \psi^\Rightarrow(\mathbb{F})$.

Now we have known that $cl_p(k\phi\mathbb{F}) \xrightarrow{q} x$ and $cl_p(k\phi\mathbb{F}) \subseteq \psi^\Rightarrow(\mathbb{F})$. Therefore, $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$, as desired.

p -(**TR**) \implies p -(**TC**). Let

$$J = \{(\mathbb{G}, y) \in \mathbb{F}_L^\top(X) \times X \mid \mathbb{G} \xrightarrow{p} y\}; \psi : J \rightarrow X, (\mathbb{G}, y) \mapsto y; \phi : J \rightarrow \mathbb{F}_L^\top(X), (\mathbb{G}, y) \mapsto \mathbb{G}.$$

Then $\forall j \in J, \phi(j) \xrightarrow{p} \psi(j)$. Please note that $j = (\mathbb{G}, y) \in J \iff \mathbb{G} = \phi(j), y = \psi(j)$.

(1) For any $\lambda, \mu \in L^X$, $S_X(\bar{\mu}_p, \lambda) = S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda))$. Indeed,

$$\begin{aligned} S_X(\bar{\mu}_p, \lambda) &= \bigwedge_{y \in X} ([\bigvee_{\mathbb{G} \xrightarrow{p} y} \Lambda(\mathbb{G})(\mu)] \rightarrow \lambda(y)) = \bigwedge_{y \in X} \bigwedge_{(\mathbb{G}, y) \in J} (\Lambda(\mathbb{G})(\mu) \rightarrow \lambda(y)) \\ &= \bigwedge_{j=(\mathbb{G}, y) \in J} (\Lambda(\phi(j))(\mu) \rightarrow \lambda(\psi(j))) = \bigwedge_{j \in J} (\hat{\phi}(\mu)(j) \rightarrow \psi^{\leftarrow}(\lambda)(j)) \\ &= S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)). \end{aligned}$$

(2) For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the family $\{\hat{\phi}(\lambda) | \lambda \in \mathbb{F}\}$ forms a \top -filter base on J . Indeed,

(TB1): For any $\lambda \in \mathbb{F}$, by $[y]_\top \xrightarrow{p} y$ for any $y \in X$, we have

$$\bigvee_{j \in J} \hat{\phi}(\lambda)(j) = \bigvee_{j \in J} \Lambda(\phi(j))(\lambda) = \bigvee_{y \in X} \bigvee_{\mathbb{G} \xrightarrow{p} y} \Lambda(\mathbb{G})(\lambda) \geq \bigvee_{y \in X} \Lambda([y]_\top)(\lambda) = \bigvee_{y \in X} \lambda(y) = \top.$$

(TB2): For any $\lambda, \mu \in \mathbb{F}$, note that for any $j \in J$,

$$\begin{aligned} \hat{\phi}(\lambda)(j) \wedge \hat{\phi}(\mu)(j) &= \bigvee_{\lambda_1 \in \phi(j)} S_X(\lambda_1, \lambda) \wedge \bigvee_{\mu_1 \in \phi(j)} S_X(\mu_1, \mu) \\ &\leq \bigvee_{\lambda_1, \mu_1 \in \phi(j)} S_X(\lambda_1 \wedge \mu_1, \lambda \wedge \mu) \\ &\leq \bigvee_{v \in \phi(j)} S_X(v, \lambda \wedge \mu) = \hat{\phi}(\lambda \wedge \mu)(j), \end{aligned}$$

i.e., $\hat{\phi}(\lambda) \wedge \hat{\phi}(\mu) \leq \hat{\phi}(\lambda \wedge \mu)$. It follows easily that (TB2) is satisfied. We denote the \top -filter generated by $\{\hat{\phi}(\lambda) | \lambda \in \mathbb{F}\}$ as \mathbb{F}^ϕ .

(3) For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $k\phi\mathbb{F}^\phi \supseteq \mathbb{F}$. Indeed, for any $\lambda \in \mathbb{F}$, we have $\hat{\phi}(\lambda) \in \mathbb{F}^\phi$, i.e., $\lambda \in k\phi\mathbb{F}^\phi$.

(4) For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $\psi^\Rightarrow(\mathbb{F}^\phi) = cl_p(\mathbb{F})$. Indeed,

$$\lambda \in \psi^\Rightarrow(\mathbb{F}^\phi) \iff \psi^{\leftarrow}(\lambda) \in \mathbb{F}^\phi \iff \bigvee_{\mu \in \mathbb{F}} S_J(\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)) = \top \stackrel{(1)}{\iff} \bigvee_{\mu \in \mathbb{F}} S_X(\bar{\mu}_p, \lambda) = \top \iff \lambda \in cl_p(\mathbb{F}).$$

Assume that $\mathbb{F} \xrightarrow{q} x$, then by (3), we have $k\phi\mathbb{F}^\phi \supseteq \mathbb{F}$, and so $k\phi\mathbb{F}^\phi \xrightarrow{q} x$. From p -(TR) and (4), we get that $cl_p(\mathbb{F}) = \psi^\Rightarrow(\mathbb{F}^\phi) \xrightarrow{q} x$. Therefore, the condition p -(TC) is satisfied. \square

The next theorem shows that p -regularity is preserved under initial structures.

Theorem 2. Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of \top -convergence spaces such that each q_i is p_i -regular. If q (resp., p) is the initial structure on X regarding the source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$ (resp., $(X \xrightarrow{f_i} (X_i, p_i))_{i \in I}$), then q is also p -regular.

Proof. Let $\psi : J \rightarrow X$ and $\phi : J \rightarrow \mathbb{F}_L^\top(X)$ be any function such that $\phi(j) \xrightarrow{p} \psi(j)$ for any $j \in J$. Then

$$\forall i \in I, \forall j \in J, (f_i^\Rightarrow \circ \phi)(j) = f_i^\Rightarrow(\phi(j)) \xrightarrow{p_i} f_i(\psi(j)) = (f_i \circ \psi)(j).$$

Let $\mathbb{F} \in \mathbb{F}_L^\top(J)$ satisfy $k\phi\mathbb{F} \xrightarrow{q} x$. Then by definition of q and Lemma 3 we have

$$\forall i \in I, k(f_i^\Rightarrow \circ \phi)\mathbb{F} = f_i^\Rightarrow(k\phi\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

Since q_i is p_i -regular we have $f_i^\Rightarrow \psi^\Rightarrow(\mathbb{F}) = (f_i \circ \psi)^\Rightarrow(\mathbb{F}) \xrightarrow{q_i} f_i(x)$. By definition of q we have $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$. Thus q is p -regular. \square

The next theorem shows that p -regularity is preserved under final structures with some additional assumptions.

Theorem 3. Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of \top -convergence spaces such that each q_i is p_i -regular. Let q (resp., p) be the final structure on X relative to the sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$ (resp., $((X_i, p_i) \xrightarrow{f_i} X)_{i \in I}$). If $X = \cup_{i \in I} f_i(X_i)$ and each $f_i : (X_i, p_i) \rightarrow (X, p)$ is a closure function, then q is also p -regular.

Proof. Let $\mathbb{F} \in \mathbb{F}_L^\top(X) \xrightarrow{q} x$. Then by definition of q , there exists $i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^\top(X_i), f_i(x_i) = x$ such that $\mathbb{G}_i \xrightarrow{q_i} x_i$ and $f_i^\rightarrow(\mathbb{G}_i) \subseteq \mathbb{F}$. Because q_i is p_i -regular we get $cl_{p_i}(\mathbb{G}_i) \xrightarrow{q_i} x_i$ and then $f_i^\rightarrow(cl_{p_i}(\mathbb{G}_i)) \xrightarrow{q} x$. By f_i is a closure function and Proposition 2 (2) it follows that $cl_p(f_i^\rightarrow(\mathbb{G}_i)) \xrightarrow{q} x$. Hence $cl_p(\mathbb{F}) \xrightarrow{q} x$ from $cl_p(f_i^\rightarrow(\mathbb{G}_i)) \subseteq cl_p(\mathbb{F})$. Thus q is p -regular. \square

For any $\{q_i\}_{i \in I} \subseteq \top(X)$, note that the supremum (resp., infimum) of $\{q_i\}_{i \in I}$ in the lattice $\top(X)$, denoted as $\sup\{q_i | i \in I\}$ (resp., $\inf\{q_i | i \in I\}$), is precisely the initial structure (resp., final structure) regarding the source $(X \xrightarrow{id_X} (X, q_i))_{i \in I}$ (resp., the sink $((X, q_i) \xrightarrow{id_X} X)_{i \in I}$). By Theorems 2 and 3, we obtain easily the following corollary. It will show us that p -regularity is preserved under supremum and infimum in the lattice $\top(X)$.

Corollary 1. Let $\{q_i | i \in I\} \subseteq \top(X)$ and $p \in \top(X)$ with each (X, q_i) being p -regular. Then both $\inf\{q_i\}_{i \in I}$ and $\sup\{q_i\}_{i \in I}$ are all p -regular.

4. Lower (Upper) p -Regular Modifications in \top -Convergence Spaces

In this section, we shall consider the p -regular modifications in \top -convergence spaces.

Lemma 6. Let p, q be \top -convergence structures on X .

- (1) If q is p -regular, then $\mathbb{F} \xrightarrow{q} x$ implies $cl_p^n(\mathbb{F}) \xrightarrow{q} x$ for any $n \in \mathbb{N}$.
- (2) If q is p -regular, then q is p' -regular for any $p \leq p'$.
- (3) The indiscrete structure ι is p -regular for any $p \in \top(X)$.

Proof. It is obvious. \square

4.1. Lower p -Regular Modification

It has been known that p -regularity is preserved under supremum in the lattice $\top(X)$ (see Corollary 1), and the indiscrete structure ι is p -regular for any $p \in \top(X)$ (see Lemma 6 (3)). So, it follows easily that for a pair of \top -convergence spaces (X, p, q) , there is a finest p -regular \top -convergence structure $\gamma_p q$ on X which is coarser than q .

Definition 7. Let (X, p, q) be a pair of \top -convergence spaces. Then the \top -convergence structure $\gamma_p q$ on X is said to be the lower p -regular modification of q .

The following theorem gives a characterization on lower p -regular modification.

Theorem 4. For any $p, q \in \top(X)$, $\mathbb{F} \xrightarrow{\gamma_p q} x \iff \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$.

Proof. We define q' as $\mathbb{F} \xrightarrow{q'} x \iff \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$, then we prove $\gamma_p q = q'$.

Obviously, $q' \in \top(X)$ and $q' \leq q$. We check that q' is p -regular. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$. It follows that $cl_p(\mathbb{F}) \supseteq cl_p(cl_p^n(\mathbb{G})) = cl_p^{n+1}(\mathbb{G})$, so $cl_p(\mathbb{F}) \xrightarrow{q'} x$. Now, we have proved that q' is p -regular.

Let r be p -regular with $r \leq q$. We prove below $r \leq q'$. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq cl_p^n(\mathbb{G})$, so $\mathbb{G} \xrightarrow{r} x$ by $q \leq r$. Because r is p -regular it follows by Lemma 6 (1) that $\mathbb{F} \supseteq cl_p^n(\mathbb{G}) \xrightarrow{r} x$. Therefore, $r \leq q'$. \square

Theorem 5. *If $f : (X, q) \rightarrow (X', q')$ and $f : (X, p) \rightarrow (X', p')$ are both continuous function between \top -convergence spaces then so is $f : (X, \gamma_p q) \rightarrow (X', \gamma_{p'} q')$.*

Proof. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $x \in X$.

$$\begin{aligned} \mathbb{F} \xrightarrow{\gamma_p q} x &\implies \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } \mathbb{F} \supseteq cl_p^n(\mathbb{G}) \\ &\implies \exists n \in \mathbb{N}, f^\Rightarrow(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^\Rightarrow(\mathbb{F}) \supseteq f^\Rightarrow(cl_p^n(\mathbb{G})) \\ &\implies \exists n \in \mathbb{N}, f^\Rightarrow(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^\Rightarrow(\mathbb{F}) \supseteq cl_{p'}^n(f^\Rightarrow(\mathbb{G})) \\ &\implies f^\Rightarrow(\mathbb{F}) \xrightarrow{\gamma_{p'} q'} (f(x)), \end{aligned}$$

where the second implication uses the continuity of $f : (X, q) \rightarrow (X', q')$, and the third implication uses the continuity of $f : (X, p) \rightarrow (X', p')$ and Proposition 2(1). \square

The following theorem exhibits us that lower p -regular modification and final structures have good compatibility.

Theorem 6. *Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of spaces in \top -CS and let q be the final structure relative to the sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$ with $X = \cup_{i \in I} f_i(X_i)$. If $p \in \top(X)$ such that each $f_i : (X_i, p_i) \rightarrow (X, p)$ is a continuous closure function, then $\gamma_p q$ is the final structure relative to the sink $((X_i, \gamma_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$.*

Proof. Let s denote the final structure relative to the sink $((X_i, \gamma_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$. Then for any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $x \in X$

$$\begin{aligned} \mathbb{F} \xrightarrow{s} x &\implies \exists i \in I, x_i \in X_i, f_i(x_i) = x, \mathbb{G}_i \xrightarrow{\gamma_{p_i} q_i} x_i \text{ s.t. } f_i^\Rightarrow(\mathbb{G}_i) \subseteq \mathbb{F}, \text{ by Theorem 4} \\ &\implies \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i(x_i) = x, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } cl_{p_i}^n(\mathbb{H}_i) \subseteq \mathbb{G}_i, f_i^\Rightarrow(\mathbb{G}_i) \subseteq \mathbb{F}, \text{ by Proposition 2 (1)} \\ &\implies \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^\Rightarrow(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } cl_p^n(f_i^\Rightarrow(\mathbb{H}_i)) \subseteq f_i^\Rightarrow(cl_{p_i}^n(\mathbb{H}_i)) \subseteq f_i^\Rightarrow(\mathbb{G}_i), f_i^\Rightarrow(\mathbb{G}_i) \subseteq \mathbb{F} \\ &\implies \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^\Rightarrow(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } cl_p^n(f_i^\Rightarrow(\mathbb{H}_i)) \subseteq \mathbb{F} \\ &\implies \mathbb{F} \xrightarrow{\gamma_p q} x. \end{aligned}$$

Conversely,

$$\begin{aligned} \mathbb{F} \xrightarrow{\gamma_p q} x &\implies \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\implies \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^\Rightarrow(\mathbb{H}_i) \subseteq \mathbb{G}, cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\implies \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } cl_{p_i}^n(f_i^\Rightarrow(\mathbb{H}_i)) \subseteq cl_p^n(\mathbb{G}), cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\implies \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^\Rightarrow(cl_{p_i}^n(\mathbb{H}_i)) \subseteq cl_p^n(f_i^\Rightarrow(\mathbb{H}_i)) \subseteq \mathbb{F} \\ &\implies \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, cl_{p_i}^n(\mathbb{H}_i) \xrightarrow{\gamma_{p_i} q_i} x_i \text{ s.t. } f_i^\Rightarrow(cl_{p_i}^n(\mathbb{H}_i)) \subseteq \mathbb{F} \\ &\implies \mathbb{F} \xrightarrow{s} x, \end{aligned}$$

where the fourth implication follows by Proposition 2(2). \square

The following corollary tells us that lower p -regular modification has good compatibility with infimum in the lattice $\top(X)$.

Corollary 2. Assume that $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ and $q = \inf\{q_i | i \in I\}$. Then $\gamma_p q = \inf\{\gamma_p q_i | i \in I\}$.

4.2. Upper p -Regular Modification

Similar to the crisp case, the discrete structure δ is not always p -regular for any $p \in \top(X)$. This shows that for a given $q \in \top(X)$, there may not exist p -regular \top -convergence structure on X finer than q .

Definition 8. Let (X, p, q) be a pair of \top -convergence spaces. If there exists a coarsest p -regular \top -convergence structure $\gamma^p q$ on X finer than q , then it is said to be the upper p -regular modification of q .

It has been known that the existence of $\gamma^p q$ depends on the existence of a p -regular \top -convergence structure finer than q (see Corollary 1), and $\gamma_p \delta$ is the finest p -regular \top -convergence structure. So, it follows easily that $\gamma^p q$ exists if and only if $q \leq \gamma_p \delta$. By Theorem 4, we obtain the following result.

Theorem 7. Let (X, p, q) be a pair of \top -convergence spaces. Then

$$\gamma^p q \text{ exists} \iff \forall x \in X, \forall n \in \mathbb{N}, cl_p^n([x]_\top) \xrightarrow{q} x.$$

Proof. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and any $x \in X$, from Theorem 4 we obtain

$$\mathbb{F} \xrightarrow{\gamma_p \delta} x \iff \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F}.$$

Necessity. Let $\gamma^p q$ exist. Then $q \leq \gamma_p \delta$. So, for any $x \in X, n \in \mathbb{N}$

$$[x]_\top \xrightarrow{\delta} x \implies cl_p^n([x]_\top) \xrightarrow{\gamma_p \delta} x \implies cl_p^n([x]_\top) \xrightarrow{q} x.$$

Sufficiency. Let $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$. Then

$$\begin{aligned} \mathbb{F} \xrightarrow{\gamma_p \delta} x &\implies \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\implies \exists n \in \mathbb{N}, [x]_\top \subseteq \mathbb{G} \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\stackrel{\text{Proposition 1 (2)}}{\implies} \exists n \in \mathbb{N}, cl_p^n([x]_\top) \subseteq cl_p^n(\mathbb{G}) \text{ s.t. } cl_p^n(\mathbb{G}) \subseteq \mathbb{F} \\ &\implies \exists n \in \mathbb{N} \text{ s.t. } cl_p^n([x]_\top) \subseteq \mathbb{F} \\ &\implies \mathbb{F} \xrightarrow{q} x. \end{aligned}$$

It follows that $q \leq \gamma_p \delta$, so $\gamma^p q$ exists. \square

The following theorem gives a characterization on upper p -regular modification if it exists.

Theorem 8. Let (X, p, q) be a pair of \top -convergence spaces and $\gamma^p q$ exists. Then

$$\mathbb{F} \xrightarrow{\gamma^p q} x \iff \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x.$$

Proof. We define q' as $\mathbb{F} \xrightarrow{q'} x \iff \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x$.

- (1) $q' \in \top(X)$. It is obvious.
- (2) $q \leq q'$. In fact, let $\mathbb{F} \xrightarrow{q'} x$ then $\mathbb{F} = cl_p^0(\mathbb{F}) \xrightarrow{q} x$.

- (3) q' is p -regular. In fact, let $\mathbb{F} \xrightarrow{q'} x$. Then for any $n \in \mathbb{N}$ it holds that $cl_p^n(cl_p(\mathbb{F})) = cl_p^{n+1}(\mathbb{F}) \xrightarrow{q} x$, which means $cl_p(\mathbb{F}) \xrightarrow{q'} x$. So, q' is p -regular.
- (4) Let r be p -regular with $q \leq r$. Then $q' \leq r$. In fact, let $\mathbb{F} \xrightarrow{r} x$ then for any $n \in \mathbb{N}$, by Proposition 6 (1) it holds that $cl_p^n(\mathbb{F}) \xrightarrow{r} x$ and so $cl_p^n(\mathbb{F}) \xrightarrow{q} x$ by $q \leq r$. That means $\mathbb{F} \xrightarrow{q'} x$.

By (1)–(4) we get that q' is the coarsest p -regular \top -convergence structure finer than q . Therefore, $\gamma^p q = q'$. \square

Theorem 9. Let $f : (X, q) \rightarrow (X', q')$ be a continuous function, and $f : (X, p) \rightarrow (X', p')$ be a closure function between \top -convergence spaces. If $\gamma^p q$ and $\gamma^{p'} q'$ exist then $f : (X, \gamma^p q) \rightarrow (X', \gamma^{p'} q')$ is continuous.

Proof. Let $\mathbb{F} \xrightarrow{\gamma^p q} x$. Then $\forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x$. Since $f : (X, q) \rightarrow (X', q')$ is a continuous function and $f : (X, p) \rightarrow (X', p')$ is a closure function it holds that

$$\forall n \in \mathbb{N}, cl_{p'}^n(f^\Rightarrow(\mathbb{F})) \supseteq f^\Rightarrow(cl_p^n(\mathbb{F})) \xrightarrow{q'} f(x),$$

so $f^\Rightarrow(\mathbb{F}) \xrightarrow{\gamma^{p'} q'} f(x)$, as desired. \square

The following theorem exhibits us that the upper p -regular modification has good compatibility with initial structures.

Theorem 10. Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of spaces in \top -CS and q be the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$. Let $p \in \top(X)$ such that each $f_i : (X, p) \rightarrow (X_i, p_i)$ is continuous closure function. If $\gamma^{p_i} q_i$ exists for all $i \in I$ then so does $\gamma^p q$, and $\gamma^p q$ is precisely the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \gamma^{p_i} q_i))_{i \in I}$.

Proof. At first, we show the existence of $\gamma^p q$. By Theorem 7, it suffices to check that $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$. In fact, by the existence of $\gamma^{p_i} q_i$ we have $cl_{p_i}^n([f_i(x)]) \xrightarrow{q_i} f_i(x)$ for any $i \in I, x \in X, n \in \mathbb{N}$. Then by each $f_i : (X, p) \rightarrow (X_i, p_i)$ being a continuous closure function it holds that

$$f_i^\Rightarrow(cl_p^n([x]_\top) = cl_{p_i}^n(f_i^\Rightarrow([x]_\top)) = cl_{p_i}^n([f_i(x)]_\top) \xrightarrow{q_i} f_i(x),$$

so $cl_p^n([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$, i.e., $\gamma^p q$ exists.

Let s denote the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \gamma^{p_i} q_i))_{i \in I}$. Then

$$\begin{aligned} \mathbb{F} \xrightarrow{s} x &\iff \forall i \in I, f_i^\Rightarrow(\mathbb{F}) \xrightarrow{\gamma^{p_i} q_i} f_i(x) \stackrel{\text{Theorem 8}}{\iff} \forall i \in I, \forall n \in \mathbb{N}, cl_{p_i}^n(f_i^\Rightarrow(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\ &\stackrel{\text{Proposition 2}}{\iff} \forall i \in I, \forall n \in \mathbb{N}, f_i^\Rightarrow(cl_p^n(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\ &\iff \forall n \in \mathbb{N}, cl_p^n(\mathbb{F}) \xrightarrow{q} x \stackrel{\text{Theorem 8}}{\iff} \mathbb{F} \xrightarrow{\gamma^p q} x. \quad \square \end{aligned}$$

The following corollary tells us that upper p -regular modification has good compatibility with supremum in the lattice $\top(X)$.

Corollary 3. Assume that $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ and $q = \sup\{q_i | i \in I\}$. If $\gamma^{p_i} q_i$ exists for all $i \in I$ then so does $\gamma^p q$ and $\gamma^p q = \sup\{\gamma^{p_i} q_i | i \in I\}$.

5. Conclusions

In this paper, we studied p -regularity in \top -convergence spaces by a diagonal condition and a closure condition about \top -filter, respectively. We proved that p -regularity was preserved under the initial and final constructions in the category \top -CS. We then followed as a conclusion that p -regularity was preserved under the infimum and supremum in the lattice $\top(X)$. Furthermore, we defined and discussed lower (upper) p -regular modifications in \top -convergence spaces. In particular, we showed that lower (resp., upper) p -regular modification has good compatibility with final (resp., initial) construction.

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References

1. Preuss, G. *Foundations of Topology*; Kluwer Academic Publishers: London, UK, 2002.
2. Kent, D.C.; Richardson, G.D. Convergence spaces and diagonal conditions. *Topol. Its Appl.* **1996**, *70*, 167–174. [[CrossRef](#)]
3. Kowalsky, H.J.; Kompletierung, L. *Math. Nachrichten* **1954**, *12*, 301–340. [[CrossRef](#)]
4. Cook, C.H.; Fischer, H.R. Regular Convergence Spaces. *Math. Annalen* **1967**, *174*, 1–7. [[CrossRef](#)]
5. Kent, D.C.; Richardson, G.D. p -regular convergence spaces. *Math. Nachrichten* **1990**, *149*, 215–222. [[CrossRef](#)]
6. Wilde, S.A.; Kent, D.C. p -topological and p -regular: Dual notions in convergence theory. *Int. J. Math. Math. Sci.* **1999**, *22*, 1–12. [[CrossRef](#)]
7. Jäger, G. A category of L -fuzzy convergence spaces. *Quaestiones Mathematicae* **2001**, *24*, 501–517. [[CrossRef](#)]
8. Flores, P.V.; Mohapatra, R.N.; Richardson, G. Lattice-valued spaces: Fuzzy convergence. *Fuzzy Sets Syst.* **2006**, *157*, 2706–2714. [[CrossRef](#)]
9. Fang, J.M. Stratified L -ordered convergence structures. *Fuzzy Sets Syst.* **2010**, *161*, 2130–2149. [[CrossRef](#)]
10. Flores, P.V.; Richardson, G. Lattice-valued convergence: Diagonal axioms. *Fuzzy Sets Syst.* **2008**, *159*, 2520–2528. [[CrossRef](#)]
11. Han, S.E.; Lu, L.X.; Yao, W. Quantale-valued fuzzy scott topology. *Iran. J. Fuzzy Syst.* **2018**, in press.
12. Jäger, G. Pretopological and topological lattice-valued convergence spaces. *Fuzzy Sets Syst.* **2007**, *158*, 424–435. [[CrossRef](#)]
13. Jäger, G. Fischer's diagonal condition for lattice-valued convergence spaces. *Quaest. Math.* **2008**, *31*, 11–25. [[CrossRef](#)]
14. Jäger, G. Gähler's neighbourhood condition for lattice-valued convergence spaces. *Fuzzy Sets Syst.* **2012**, *204*, 27–39. [[CrossRef](#)]
15. Jäger, G. Stratified LMN-convergence tower spaces. *Fuzzy Sets Syst.* **2016**, *282*, 62–73. [[CrossRef](#)]
16. Jin, Q.; Li, L.Q. Stratified lattice-valued neighborhood tower group. *Quaest. Math.* **2018**, *41*, 847–861. [[CrossRef](#)]
17. Jin, Q.; Li, L.Q.; Lv, Y.R.; Zhao, F.; Zhou, J. Connectedness for lattice-valued subsets in lattice-valued convergence spaces. *Quaest. Math.* **2019**, *42*, 135–150. [[CrossRef](#)]
18. Jin, Q.; Li, L.Q.; Meng, G.W. On the relationships between types of L -convergence spaces. *Iran. J. Fuzzy Syst.* **2016**, *1*, 93–103.
19. Li, L.Q.; Jin, Q. On adjunctions between Lim , $SL\text{-Top}$, and $SL\text{-Lim}$. *Fuzzy Sets Syst.* **2011**, *182*, 66–78. [[CrossRef](#)]
20. Li, L.Q.; Jin, Q. On stratified L -convergence spaces: Pretopological axioms and diagonal axioms. *Fuzzy Sets Syst.* **2012**, *204*, 40–52. [[CrossRef](#)]
21. Li, L.Q.; Jin, Q.; Hu, K. On Stratified L -Convergence Spaces: Fischer's Diagonal Axiom. *Fuzzy Sets Syst.* **2015**, *267*, 31–40. [[CrossRef](#)]

22. Losert, B.; Boustique, H.; Richardson, G. Modifications: Lattice-valued structures. *Fuzzy Sets Syst.* **2013**, *210*, 54–62. [[CrossRef](#)]
23. Orpen, D.; Jäger, G. Lattice-valued convergence spaces: Extending the lattices context. *Fuzzy Sets Syst.* **2012**, *190*, 1–20. [[CrossRef](#)]
24. Pang, B.; Zhao, Y. Several types of enriched (L, M) -fuzzy convergence spaces. *Fuzzy Sets Syst.* **2017**, *321*, 55–72. [[CrossRef](#)]
25. Pang, B. Degrees of Separation Properties in Stratified L -Generalized Convergence Spaces Using Residual Implication. *Filomat* **2017**, *31*, 6293–6305. [[CrossRef](#)]
26. Pang, B. Stratified L -ordered filter spaces. *Quaest. Math.* **2017**, *40*, 661–678. [[CrossRef](#)]
27. Pang, B.; Xiu, Z.Y. Stratified L -prefilter convergence structures in stratified L -topological spaces. *Soft Comput.* **2018**, *22*, 7539–7551. [[CrossRef](#)]
28. Xiu, Z.Y.; Pang, B. Base axioms and subbase axioms in M -fuzzifying convex spaces. *Iran. J. Fuzzy Syst.* **2018**, *15*, 75–87.
29. Yang, X.; Li, S. Net-theoretical convergence in (L, M) -fuzzy cotopological spaces. *Fuzzy Sets Syst.* **2012**, *204*, 53–65. [[CrossRef](#)]
30. Yao, W. On many-valued stratified L -fuzzy convergence spaces. *Fuzzy Sets Syst.* **2008**, *159*, 2503–2519. [[CrossRef](#)]
31. Fang, J.M.; Yue, Y.L. \top -diagonal conditions and Continuous extension theorem. *Fuzzy Sets Syst.* **2017**, *321*, 73–89. [[CrossRef](#)]
32. Jin, Q.; Li, L.Q. Modified Top-convergence spaces and their relationships to lattice-valued convergence spaces. *J. Intell. Fuzzy Syst.* **2018**, *35*, 2537–2546. [[CrossRef](#)]
33. Li, L.Q. p -Topologicalness—A Relative Topologicalness in \top -Convergence Spaces. *Mathematics* **2019**, *7*, 228. [[CrossRef](#)]
34. Li, L.Q.; Jin, Q.; Hu, K. Lattice-valued convergence associated with CNS spaces. *Fuzzy Sets Syst.* **2018**. [[CrossRef](#)]
35. Qiu, Y.; Fang, J.M. The category of all \top -convergence spaces and its cartesian-closedness. *Iran. J. Fuzzy Syst.* **2017**, *14*, 121–138.
36. Reid, L.; Richardson, G. Connecting \top and Lattice-Valued Convergences. *Iran. J. Fuzzy Syst.* **2018**, *15*, 151–169.
37. Jäger, G. Lattice-valued convergence spaces and Regularity. *Fuzzy Sets Syst.* **2008**, *159*, 2488–2502. [[CrossRef](#)]
38. Boustique, H.; Richardson, G. A note on regularity. *Fuzzy Sets Syst.* **2011**, *162*, 64–66. [[CrossRef](#)]
39. Boustique, H.; Richardson, G. Regularity: Lattice-valued Cauchy spaces. *Fuzzy Sets Syst.* **2012**, *190*, 94–104. [[CrossRef](#)]
40. Li, L.Q.; Jin, Q. p -Topologicalness and p -Regularity for lattice-valued convergence spaces. *Fuzzy Sets Syst.* **2014**, *238*, 26–45. [[CrossRef](#)]
41. Li, L.Q.; Jin, Q.; Meng, G.W.; Hu, K. The lower and upper p -topological (p -regular) modifications for lattice-valued convergence spaces. *Fuzzy Sets Syst.* **2016**, *282*, 47–61. [[CrossRef](#)]
42. Li, L.Q.; Jin, Q.; Yao, B.X. Regularity of fuzzy convergence spaces. *Open Math.* **2018**, *16*, 1455–1465. [[CrossRef](#)]
43. Höhle, U.; Šostak, A. Axiomatic foundations of fixed-basis fuzzy topology. In *Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series*; Höhle, U., Rodabaugh, S.E., Eds.; Kluwer Academic Publishers: Boston, MA, USA; Dordrecht, The Netherlands; London, UK, 1999; Volume 3, pp. 123–273.
44. Bělohlávek, R. *Fuzzy Relational Systems: Foundations and Principles*; Kluwer Academic Publishers: New York, NY, USA, 2002.
45. Zhang, D.X. An enriched category approach to many valued Topology. *Fuzzy Sets Syst.* **2007**, *158*, 349–366. [[CrossRef](#)]
46. Zhao, F.F.; Jin, Q.; Li, L.Q. The axiomatic characterizations on L -generalized fuzzy neighborhood system-based approximation operators. *Int. J. Gen. Syst.* **2018**, *47*, 155–173. [[CrossRef](#)]
47. Lai, H.; Zhang, D. Fuzzy topological spaces with conical neighborhood system. *Fuzzy Sets Syst.* **2018**, *330*, 87–104. [[CrossRef](#)]

48. García, J.G. On stratified L -valued filters induced by \top -filters. *Fuzzy Sets Syst.* **2006**, *157*, 813–819. [[CrossRef](#)]
49. Adámek, J.; Herrlich, H.; Strecker, G.E. *Abstract and Concrete Categories*; Wiley: New York, NY, USA, 1990.



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