mathematics

MDPI

Article

A Kind of New Higher-Order Mond-Weir Type Duality for Set-Valued Optimization Problems

Liu He [1,†], Qi-Lin Wang [1,*], Ching-Feng Wen [2,3,*], Xiao-Yan Zhang [1] and Xiao-Bing Li [1]

1 College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China; heliu20170906@163.com(L.H.); zhangxy4732@163.com(X.-Y.Z.); xiaobinglicq@126.com(X.-B.L.)
2 Center for Fundamental Science; and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan
3 Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 80708, Taiwan
* Correspondence: wangql97@126.com(Q.-L.W.); cfwen@kmu.edu.tw (C.-F.W.)
† Current address: No.66 Xuefu Rd., Nan'an Dist., Chongqing 400074, China.

Received: 9 March 2019; Accepted: 17 April 2019; Published: 24 April 2019

check for updates

Abstract: In this paper, we introduce the notion of higher-order weak adjacent epiderivative for a set-valued map without lower-order approximating directions and obtain existence theorem and some properties of the epiderivative. Then by virtue of the epiderivative and Benson proper efficiency, we establish the higher-order Mond-Weir type dual problem for a set-valued optimization problem and obtain the corresponding weak duality, strong duality and converse duality theorems, respectively.

Keywords: set-valued optimization problems; higher-order weak adjacent epiderivatives; higher-order mond-weir type dual; benson proper efficiency

1. Introduction

The theory of duality and optimality conditions for optimization problems has received considerable attention (see [1–10]). The derivative (epiderivative) plays an important role in studying duality and optimality conditions for set-valued optimization problems. The contingent derivatives [1], the contingent epiderivatives [11] and the generalized contingent epiderivatives [12] for set-valued maps are employed by different authors to investigate necessary or/and sufficient optimality conditions for set-valued optimization problems. Later, the second-order epiderivatives [13], higher-order generalized contingent (adjacent) epiderivatives [14] and generalized higher-order contingent (adjacent) derivatives [15] for set-valued maps are used to study the second (or high) order necessary or/and sufficient optimality conditions for set-valued optimization problems. Chen et al. [2] utilized the weak efficiency to introduce higher-order weak adjacent (contingent) epiderivative for a set-valued map, they then investigate higher-order Mond-Weir (Wolfe) type duality and higher-order Kuhn-Tucker type optimality conditions for constrained set-valued optimization problems. Li et al. [3] used the higher-order contingent derivatives to discuss the weak duality, strong duality and converse duality of a higher-order Mond-Weir type dual for a set-valued optimization problem. Wang et al. [4] used the higher-order generalized adjacent derivative to extend the main results of [3] from convexity to non-convexity. Anh [6] used the higher-order radial derivatives [16] to discuss mixed duality of set-valued optimization problems.

It is well known that the lower-order approximating directions are very important to define the higher-order derivatives (epiderivatives) in [2–4,6,14,15]. This limits their practical applications when the lower-order approximating directions are unknown. So, it is necessary to introduce some higher-order derivatives (epiderivatives) without lower-order approximating directions. As we know, a few paper are devoted to this topic. Motivated by [17], Li et al. [7] proposed the higher-order upper and lower Studniarski derivatives of a set-valued map to establish necessary and sufficient conditions for a strict local minimizer of a constrained set-valued optimization problem. Anh [8] introduced the higher-order radial epiderivative to establish mixed type duality in constrained set-valued optimization problems. Anh [18] proposed the higher-order upper and lower Studniarski derivatives of a set-valued map to establish Fritz John type and Kuhn-Tucker type conditions, and discussed the higher-order Mond-Weir type dual for constrained set-valued optimization problems. Anh [19] further defined the notion of higher-order Studniarski epiderivative and established higher-order optimality conditions for a generalized set-valued optimization problems. Anh [20] noted that the epiderivatives in [8,19] is singleton, they proposed a notion of the higher-order generalized Studniarski epiderivative which is set-valued, and discussed its applications in optimality conditions and duality of set-valued optimization problems.

As we know that the existence conditions of weak efficient point are weaker than ones of efficient point for a set. Inspired by [2,8,18–20], we introduce the higher-order weak adjacent set without the lower-order approximating directions for set-valued maps. Furthermore, we use the higher-order weak adjacent set and weak efficiency to introduce the higher-order weak adjacent epiderivative for a set-valued map, we use it and Benson proper efficiency to discuss higher-order Mond-Weir type dual for a constrained set-valued optimization problem, and then obtain the corresponding weak duality, strong duality and converse duality, respectively.

The rest of the article is as follows. In Section 2, we recall some of definitions and notations to be needed in the paper, and so define the higher-order adjacent set of a set-valued map without lower-order approximating directions, which has some nice properties. In Section 3, we use the higher-order adjacent set of Section 2 to define the higher-order weak adjacent epiderivative for a set-valued map, and discuss its properties, such as existence and subdifferential. In Section 4, we introduce a higher-order Mond-Weir type dual for a constrained set-valued optimization problem and establish the corresponding weak duality, strong duality and converse duality, respectively.

2. Preliminaries

Throughout the paper, let X , Y and Z be three real normed linear spaces. The spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subseteq Y$ and $D \subseteq Z$ with nonempty interior, respectively. By 0_Y we denote the zero vector of Y. Y^* stands for the topological dual space of Y. The dual cone C^+ of C is defined as

$$C^+ := \{f \in Y^* | f(c) \geq 0, \forall c \in C\}.$$

Its quasi-interior C^{+i} is defined as

$$C^{+i} := \{f \in Y^* | f(c) > 0, \forall c \in C \setminus \{0_Y\}\}.$$

Let M be a nonempty subset of Y. We denote the closure, the interior and the cone hull of M by $\mathrm{cl}M$, $\mathrm{int}M$ and $\mathrm{cone}M$, respectively. We denote by $B(c, r)$ the open ball of radius r centered at c. A nonempty subset B of C is called a base of C if and only if $C = \mathrm{cone}B$ and $0_Y \notin \mathrm{cl}B$.

Let $E \subseteq X$ be a nonempty subset and $F : E \to 2^Y$ be a set-valued map. The domain, graph and epigraph of F are, respectively, defined as

$$\mathrm{dom}F := \{x \in E | F(x) \neq \emptyset\}, \quad \mathrm{im}F := \{y \in Y | y \in F(x)\},$$

$$\mathrm{gr}F := \{(x, y) \in E \times Y | y \in F(x), x \in E\}$$

and

$$\mathrm{epi}F := \{(x, y) \in E \times Y | y \in F(x) + C, x \in E\}.$$

Definition 1. *[9] Let $M \subseteq Y$ and $y_0 \in M$.*
(i) y_0 is said to be a Pareto efficient point of M ($y_0 \in \mathrm{Min}_C M$) if

$$(M - \{y_0\}) \cap (-C \setminus \{0_Y\}) = \emptyset.$$

(ii) Let $\mathrm{intC} \neq \emptyset$. y_0 is said to be a weakly efficient point of M ($y_0 \in \mathrm{WMin}_C M$) if

$$(M - \{y_0\}) \cap (-\mathrm{intC}) = \emptyset.$$

Definition 2. *[10,21,22] (i) The cone C is called Daniell if any decreasing sequence in Y that has a lower bound converges to its infimum.*

(ii) A subset M of Y is said to be minorized if there is a $y \in Y$ such that

$$M \subseteq \{y\} + C.$$

(iii) The weak domination property is said to hold for a subset M of Y if

$$M \subseteq \mathrm{WMin}_C M + \mathrm{intC} \cup \{0_Y\}.$$

Definition 3. *Let $A \subseteq X \times Y$, $(x_0, y_0) \in \mathrm{cl}A$ and $m \in N \setminus \{0\}$.*
(i) [9] The mth-order adjacent set of A at $(x, v_1, \cdots, v_{m-1})$ is defined by

$$\begin{aligned} T_A^{\flat(m)}(x, v_1, \cdots, v_{m-1}) :=& \{y \in A | \forall t_n \to 0^+, \exists y_n \to y, s.t. \\ & x_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m y_n \in A\}, \end{aligned}$$

where $v_i \in X (i = 1, \cdots, m-1)$.
(ii) [19] The mth-order Studniarski set of A at (x_0, y_0) is defined by

$$\begin{aligned} S_A^m(x_0, y_0) := & \{(x, y) \in X \times Y | \exists t_n \to 0^+, \exists (x_n, y_n) \to (x, y), \\ & s.t. (x_0 + t_n x_n, y_0 + t_n^m y_n) \in A\}. \end{aligned}$$

Definition 4. *Let $K \subseteq X \times Y$, $(x_0, y_0) \in \mathrm{cl}K$ and $m \in N \setminus \{0\}$. The mth-order adjacent set of K at (x_0, y_0) is defined by*

$$\begin{aligned} T_K^{\flat(m)}(x_0, y_0) := & \{(x, y) \in X \times Y | \forall t_n \to 0^+, \exists (x_n, y_n) \to (x, y), \\ & s.t. (x_0 + t_n x_n, y_0 + t_n^m y_n) \in K\}. \end{aligned}$$

We can obtain the equivalent characterization of $T_K^{\flat(m)}(x_0, y_0)$ in terms of sequences: $(x, y) \in T_K^{\flat(m)}(x_0, y_0)$ if and only if $\forall \{t_n\} \to 0^+$, $\exists \{(x'_n, y'_n)\} \subseteq K$ such that

$$\lim_{n \to \infty} \left(\frac{x'_n - x_0}{t_n}, \frac{y'_n - y_0}{t_n^m}\right) = (x, y).$$

Now, we establish a few properties of $T_K^{\flat(m)}(x_0, y_0)$.

Proposition 1. *Let $K \subseteq X \times Y$, $(x_0, y_0) \in K$ and $(x, y) \in T_K^{\flat(m)}(x_0, y_0)$. Then*

$$(\lambda x, \lambda^m y) \in T_K^{\flat(m)}(x_0, y_0), \forall \lambda \geq 0.$$

Proof. We divide λ into two cases to show the proposition.

Case 1: $\lambda = 0$. Note that $(x_0, y_0) \in K$; for any sequence $\{t_n\}$ with $t_n \to 0^+$, we choose $(x_n, y_n) = (0_X, 0_Y)$ such that $(x_0 + t_n x_n, y_0 + t_n^m y_n) \in K$. This means that $(0_X, 0_Y) \in T_K^{\flat(m)}(x_0, y_0)$.

Case 2:$\lambda > 0$. Let $(x,y) \in T_K^{\flat(m)}(x_0,y_0)$. Then for any sequence $\{t_n\}$ with $t_n \to 0^+$, there exists a sequence $\{(x_n,y_n)\} \subseteq K$ with $(x_n,y_n) \to (x,y)$ such that

$$K \ni (x_0 + t_n x_n, y_0 + t_n^m y_n) = (x_0 + (\frac{t_n}{\lambda})\lambda x_n, y_0 + (\frac{t_n}{\lambda})^m \lambda^m y_n).$$

Naturally, $\frac{t_n}{\lambda} \to 0^+$ and $(\lambda x_n, \lambda^m y_n) \to (\lambda x, \lambda^m y) \in T_K^{\flat(m)}(x_0,y_0)$. It completes the proof. □

Remark 1. *Let $K \subseteq X \times Y$ and $(x_0,y_0) \in \mathrm{cl}K$. The mth-order adjacent set $T_K^{\flat(m)}(x_0,y_0)$ of K at (x_0,y_0) may not be a cone; see Example 1.*

Example 1. *Let $K = \{(x,y) \in \mathbb{R}^2 | y \geq x^4, x \in \mathbb{R}\}$, $(x_0,y_0) = (0,0)$ and $m = 4$. A simple calculation shows that*

$$T_K^{\flat(4)}(0,0) = \{(x,y) \in \mathbb{R}^2 | y \geq x^4\}.$$

Take $(x,y) = (1,1) \in T_K^{\flat(4)}(0,0)$ and $\lambda = 2$. Then $\lambda(x,y) = (2,2) \notin T_K^{\flat(4)}(0,0)$, i.e., $T_k^{\flat(4)}(0,0)$ is not a cone here.

Proposition 2. *Let $F : E \to 2^Y$ be a set-valued map and $(x_0,y_0) \in \mathrm{gr}F$. Then,*
(i) $T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0) = T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0) + \{0_X\} \times C$;
(ii) $\{y \in Y | (x,y) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0)\} = \{y \in Y | (x,y) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0)\} + C, \forall x \in X$.

Proof. Since $0_Y \in C$, it is clearly that $T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0) \subseteq T_{\mathrm{epi}F}^{b(m)}(x_0,y_0) + \{0_X\} \times C$. Therefore we only need to prove $T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0) + \{0_X\} \times C \subseteq T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0)$.

Let $(u,v) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0)$ and $c \in C$. Then for any sequence $\{t_n\}$ with $t_n \to 0^+$, there exists a sequence $\{(u_n,v_n)\} \subseteq X \times Y$ with $(u_n,v_n) \to (u,v)$ such that

$$(x_0 + t_n u_n, y_0 + t_n^m v_n) \in \mathrm{epi}F,$$

namely,

$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n) + C.$$

Since $c \in C$, $t_n \to 0^+$ and $C + C \subseteq C$, one has

$$y_0 + t_n^m (v_n + c) \in F(x_0 + t_n u_n) + C + \{t_n^m c\} \subseteq F(x_0 + t_n u_n) + C.$$

Thus

$$(x_0 + t_n u_n, y_0 + t_n^m (v_n + c)) \in \mathrm{epi}F.$$

This together with $(u_n, v_n + c) \to (u, v + c)$ implies $(u, v + c) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0,y_0)$, and so $T_{\mathrm{epi}F}^{\flat(m)} + \{0_X\} \times C \subseteq T_{\mathrm{epi}F}^{\flat(m)}$.

(ii) Obviously, (ii) follows from (i). The proof is complete. □

Proposition 3. *Let $K \subseteq X \times Y$ and $(x_0,y_0) \in \mathrm{cl}K$. If K is a convex set, then $T_K^{\flat(m)}(x_0,y_0)$ is a convex set.*

Proof. Let $(x^i,y^i) \in T_K^{\flat(m)}(x_0,y_0)$ $(i = 1,2)$ and $\lambda \in [0,1]$. Then for any $t_n \to 0^+$, there exist $(x_n^i, y_n^i) \to (x^i,y^i)$ $(i = 1,2)$ such that

$$(x_0 + t_n x_n^i, y_0 + t_n^m y_n^i) \in K \ (i = 1,2).$$

From the convexity of K, we have

$$(x_0 + t_n[(1-\lambda)x_n^1 + \lambda x_n^2], y_0 + t_n^m[(1-\lambda)y_n^1 + \lambda y_n^2]) \in K.$$

It is obvious that

$$((1-\lambda)x_n^1 + \lambda x_n^2, (1-\lambda)y_n^1 + \lambda y_n^2) \to ((1-\lambda)x^1 + \lambda x^2, (1-\lambda)y^1 + \lambda y^2).$$

It follows from the definition of $T_K^{\flat(m)}(x_0, y_0)$ that

$$(1-\lambda)(x^1, y^1) + \lambda(x^2, y^2) = ((1-\lambda)x^1 + \lambda x^2, (1-\lambda)y^1 + \lambda y^2) \in T_K^{\flat(m)}(x_0, y_0).$$

Thus, $T_K^{\flat(m)}(x_0, y_0)$ is a convex set and the proof is complete. □

3. Higher-Order Weak Adjacent Epiderivatives

In this section, we introduce the notion of higher-order weak adjacent epiderivative of a set-valued map without lower-order approximating directions, and obtain some properties of the epiderivative.

Firstly, we recall the notions of mth-order weak adjacent epiderivative with lower-order approximating directions and generalized Studniarski epiderivative without lower-order approximating directions.

Definition 5. *[2] Let $F: X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $(u_i, v_i) \in X \times Y (i = 1, \cdots, m-1)$. The mth-order weak adjacent epiderivative $D_w^{\flat(m)}F(x_0, y_0, u_1, v_1 \cdots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) for vectors $(u_1, v_1), \cdots,$ (u_{m-1}, v_{m-1}) is the set-valued map from X to Y defined by*

$$\begin{aligned} &D_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \\ :=&\mathrm{WMin}_C\{y \in Y \mid (x, y) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}. \end{aligned}$$

Definition 6. *[20] Let $F: X \to 2^Y$ and $(x_0, y_0) \in \mathrm{gr}F$. The mth-order generalized Studniarski epiderivative $G\text{-}ED_S^m F(x_0, y_0)$ of F at (x_0, y_0) is the set-valued map from X to Y defined by*

$$G\text{-}ED_S^m F(x_0, y_0)(x) := \mathrm{Min}_C\{y \in Y \mid (x, y) \in S_{\mathrm{epi}F}^m(x_0, y_0)\}.$$

Motivated by Definitions 5 and 6, we introduce the higher-order epiderivative without lower-order approximating directions.

Definition 7. *Let $F: E \to 2^Y$ and $(x_0, y_0) \in \mathrm{gr}F$. The mth-order weak adjacent epiderivative of F at (x_0, y_0) is a set-valued map $ED_w^{\flat(m)}F(x_0, y_0): E \to 2^Y$ defined by*

$$ED_w^{\flat(m)}F(x_0, y_0)(x) := \mathrm{WMin}_C\{y \in Y \mid (x, y) \in T_{\mathrm{epi}F}^{\flat(m)}(x_0, y_0)\}.$$

Remark 2. *There are many examples show that $ED_w^{\flat(m)}F(x_0, y_0)$ possibly exists even if $D_w^{\flat(m)}F(x_0, y_0, u_1, v_1,$ $\cdots, u_{m-1}, v_{m-1})$ and $G\text{-}ED_S^m F(x_0, y_0)$ do not; see Examples 2 and 3. Therefore it is interesting to study this derivative and employ it to investigate the Mond-Weir duality for set-valued optimization problems.*

Example 2. *Let $E = X = Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F: E \to 2^Y$ be defined by $F(x) := \{y \in Y \mid y \geq x^2\}$. Take $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$ and $(u, v) = (1, -1)$. Then, simple calculations show that*

$$T_{\mathrm{epi}F}^{\flat(2)}((0,0),(1,-1)) = \emptyset$$

and

$$T^{\flat(2)}_{\mathrm{epi}F}(0,0) = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x \in \mathbb{R}, y \geq x^2\}.$$

So, for any $x \in E$, $ED^{\flat(2)}_{w}F\big((0,0),(1,-1)\big)(x) = \emptyset$, *but* $ED^{\flat(2)}_{w}F(0,0)(x) = \{x^2\}$.

Example 3. *Let* $E = X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ *and* $F : E \to 2^Y$ *be defined by* $F(x) := \{(y_1,y_2) \in Y \mid y_1 \in \mathbb{R}, y_2 \geq x^2\}$. *Take* $(x_0,y_0) = (0_X,0_Y) \in \mathrm{gr}F$. *Then*

$$S^2_{\mathrm{epi}F}(0_X,0_Y) = T^{\flat(2)}_{\mathrm{epi}F}(0_X,0_Y) = \{(x,(y_1,y_2)) \in \mathbb{R}\times\mathbb{R}^2 \mid x \in \mathbb{R}, y_1 \in \mathbb{R}, y_2 \geq x^2\}.$$

Therefore, for any $x \in E$, $ED^{\flat(2)}_{w}F(0_X,0_Y)(x) = \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \in \mathbb{R}, y_2 = x^2\}$, *but* $G\text{-}ED^2_S F(0_X,0_Y)(x) = \emptyset$.

Theorem 1. *Let* $F : E \to 2^Y$ *and* $(x_0,y_0) \in \mathrm{gr}F$. *Let* C *be a pointed closed convex cone and Daniell. If* $P(x) := \{y \in Y \mid (x,y) \in T^{\flat(m)}_{\mathrm{epi}F}(x_0,y_0)\}$ *is minorized for all* $x \in \mathrm{dom}P$, *then* $ED^{\flat(m)}_{w}F(x_0,y_0)$ *exists.*

Proof. The proof is similar to that of Theorem 3.1 in [2]. □

Definition 8. *[23] Let* $M \subseteq \mathbb{R}^n$ *be a nonempty set and* $x_0 \in M$. M *is called star-shaped at* x_0, *if for any point* $x \in M$ *with* $x \neq x_0$, *the segment*

$$[x,x_0] := \{y \in M \mid y = (1-\lambda)x_0 + \lambda x, 0 \leq \lambda \leq 1\} \subseteq M.$$

Definition 9. *[10] Let* E *be a nonempty convex set. The map* F *is said to be* C*-convex on* E, *if for any* $x_1, x_2 \in E$ *and* $\lambda \in [0,1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C.$$

Motivated by Definition 9, we introduce the following concept.

Definition 10. *Let* E *be a star-shaped set at* $x_0 \in E$. *The map* F *is said to be generalized* C*-convex at* x_0 *on* E, *if for any* $x \in E$ *and* $\lambda \in [0,1]$,

$$(1-\lambda)F(x_0) + \lambda F(x) \subseteq F((1-\lambda)x_0 + \lambda x) + C.$$

Remark 3. *Let* E *be a convex set and* $x_0 \in E$. *If* F *is* C*-convex on* E, *then* F *is generalized* C*-convex at* x_0 *on* E. *However, the converse implication is not true.*

To understand Remark 3, we give the following example.

Example 4. *Let* $E_1 = (-\infty,-1] \subseteq \mathbb{R}$, $E_2 = (-1,1] \subseteq \mathbb{R}$, $E = X = E_1 \cup E_2 \subseteq \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ *and* $F : E \to 2^Y$ *be defined by*

$$F(x) = \begin{cases} \{y \in Y \mid y \geq 1\}, x \in E_1, \\ \{y \in Y \mid y \geq x^2\}, x \in E_2. \end{cases}$$

Take $x_0 = -1 \in E$. *Then* E *is a convex set, and* F *is generalized* C*-convex at* x_0 *on* E. *Take* $x_1 = -4 \in E_1 \subseteq E$, $x_2 = 0 \in E_2 \subseteq E$ *and* $\lambda = \frac{1}{2}$, *then*

$$\frac{1}{2}F(x_1) + \frac{1}{2}F(x_2) = \{y|y \geq \frac{1}{2}\}$$

and

$$F(\frac{1}{2}x_1 + \frac{1}{2}x_2) = \{y|y \geq 1\}.$$

Thus

$$\frac{1}{2}F(x_1)+(1-\frac{1}{2})F(x_2)\not\subseteq F(\frac{1}{2}x_1+(1-\frac{1}{2})x_2)+C.$$

Therefore F is not C-convex on E.

Definition 11. *[24] Let $U\subseteq X$ be a star-shaped set at $x_0\in U$. A set-valued map $F:U\to 2^Y$ is said to be decreasing-along-rays at x_0 if for any $x\in U$ and $0\leq t_1\leq t_2$ with $t_ix+(1-t_i)x_0\in U(i=1,2)$, one has*

$$F(t_1x+(1-t_1)x_0)\subseteq F(t_2x+(1-t_2)x_0)+C.$$

Next, we give an important property of the mth-order weak adjacent epiderivative.

Proposition 4. *Let E be a star-shaped set at $x_0\in E$. Let $F:E\to 2^Y$ be a set-valued map and $(x_0,y_0)\in \mathrm{gr}F$. Suppose that the following conditions are satisfied:*

(i) F is decreasing-along-rays at x_0 ;

(ii) F is generalized C-convex at x_0 on E;

(iii) the set $P(x):=\{y\in Y\mid (x,y)\in T^{\flat(m)}_{\mathrm{epi}F}(x_0,y_0)\}$ fulfills the weak domination property for all $x\in \mathrm{dom}P$.

Then for all $x\in E$, one has $x-x_0\in\Omega:=\mathrm{dom}ED^{\flat(m)}_wF(x_0,y_0)$ and

$$F(x)-\{y_0\}\subseteq ED^{\flat(m)}_wF(x_0,y_0)(x-x_0)+C.$$

Proof. Let $x\in E$ and $y\in F(x)$. For any $\lambda_n\in(0,1)$ with $\lambda_n\to 0^+$, $(\frac{\lambda_n}{2})^m\leq\frac{\lambda_n}{2}$. Since E is a star-shaped set at x_0,

$$x_n:=x_0+\frac{\lambda_n}{2}(x-x_0)=(1-\frac{\lambda_n}{2})x_0+\frac{\lambda_n}{2}x\in E$$

and

$$x_0+(\frac{\lambda_n}{2})^m(x-x_0)=(1-(\frac{\lambda_n}{2})^m)x_0+(\frac{\lambda_n}{2})^mx\in E.$$

Together this with conditions (i) and (ii) implies

$$\begin{aligned}y_n:&=y_0+(\frac{\lambda_n}{2})^m(y-y_0)=(1-(\frac{\lambda_n}{2})^m)y_0+(\frac{\lambda_n}{2})^my\\&\in(1-(\frac{\lambda_n}{2})^m)F(x_0)+(\frac{\lambda_n}{2})^mF(x)\subseteq F((1-(\frac{\lambda_n}{2})^m)x_0+(\frac{\lambda_n}{2})^mx)+C\\&\subseteq F(x_n)+C+C\subseteq F(x_n)+C.\end{aligned}$$

Hence, $(x_n,y_n)\in\mathrm{epi}F$. It follows from the definition of $T^{\flat(m)}_K(x_0,y_0)$ that $(x-x_0,y-y_0)\in T^{\flat(m)}_{\mathrm{epi}F}(x_0,y_0)$. Replacing $x-x_0\in\mathrm{dom}P$ with x of condition (iii), from the definition of $ED^{\flat(m)}_wF(x_0,y_0)$, we have

$$\begin{aligned}P(x-x_0)&\subseteq ED^{\flat(m)}_wF(x_0,y_0)(x-x_0)+\mathrm{int}C\cup\{0_Y\}\\&\subseteq ED^{\flat(m)}_wF(x_0,y_0)(x-x_0)+C.\end{aligned}$$

Thus $x-x_0\in\Omega$ and

$$F(x)-\{y_0\}\subseteq ED^{\flat(m)}_wF(x_0,y_0)(x-x_0)+C.$$

This completes the proof. □

We now give an example to explain Proposition 4.

Example 5. *Let $E = [0, +\infty) \subseteq \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : E \to 2^Y$ be defined as $F(x) = \{y \in Y \mid y \geq 0\}$. Take $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$. Then, simple calculations show that $T^{\flat(2)}_{\mathrm{epi}F}(0, 0) = \mathbb{R}^2_+$ and*

$$ED^{\flat(2)}_w F(0, 0)(x - x_0) = \{0\}, \forall x \geq 0.$$

We can easily see that all conditions of Proposition 4 are satisfied. For any $x \in E$, one has $x - 0 \in \Omega := \mathrm{dom}ED^{\flat(2)}_w F(0, 0) = \{x \mid x \geq 0\}$ and

$$F(x) - \{y_0\} \subseteq ED^{\flat(2)}_w F(0, 0)(x - x_0) + C.$$

Therefore Proposition 4 is applicable here.

The following examples show that every condition of Proposition 4 is necessary.

Example 6. *Let $E = [0, +\infty) \subseteq \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : E \to 2^Y$ be a set-valued map satisfing $F(x) = \{y \in Y \mid y \geq x\}$. Take $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$. By a simple calculation, we obtain*

$$T^{\flat(2)}_{\mathrm{epi}F}(0, 0) = \{(0, y) \in \mathbb{R} \times \mathbb{R} | y \geq 0\}$$

and

$$ED^{\flat(2)}_w F(0, 0)(x) = \begin{cases} \{0\}, x = 0, \\ \ \emptyset \ \ , x \neq 0. \end{cases}$$

Thus $x - 0 \notin \Omega := \mathrm{dom}ED^{\flat(2)}_w F(0, 0) = \{0\}$, for any $x \in (0, +\infty)$.
Obviously, the conditions (ii) and (iii) of Proposition 4 are satisfied except condition (i), and

$$F(x) - \{y_0\} \not\subseteq ED^{\flat(2)}_w F(x_0, y_0)(x - x_0) + C, x \in (0, +\infty).$$

Thus Proposition 4 does not hold here and the condition (i) of Proposition 4 is essential.

Example 7. *Let $E_1 = [0, 1] \subseteq \mathbb{R}, E_2 = (1, +\infty) \subseteq \mathbb{R}, E = X = E_1 \cup E_2 \subseteq \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : E \to 2^Y$ be given by*

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \geq -x^2\}, x \in E_1, \\ \{y \in \mathbb{R} \mid y \geq -x^3\}, x \in E_2. \end{cases}$$

Take $(x_0, y_0) = (0, 0) \in \mathrm{gr}F = \mathrm{epi}F$. Then,

$$T^{\flat(2)}_{\mathrm{epi}F}(0, 0) = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y \geq -x^2, x \geq 0\}$$

and

$$ED^{\flat(2)}_w F(0, 0)(x - x_0) = \{y \in \mathbb{R} \mid y = -x^2\}, \forall x \geq 0.$$

Clearly, the conditions (i) and (iii) of Proposition 4 are satisfied except condition (ii), and for any $x \in E_2$,

$$F(x) - \{y_0\} \not\subseteq ED^{\flat(2)}_w F(x_0, y_0)(x - x_0) + C.$$

Therefore Proposition 4 does not hold here and the condition (ii) of Proposition 4 is essential.

Example 8. *Let $E = X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $F : E \to 2^Y$ be defined by $F(x) := \{(y_1, y_2) \in Y \mid y_1 \in \mathbb{R}, y_2 \geq 0\}$. Take $(x_0, y_0) = (0, (0, 1)) \in \mathrm{gr}F$. Then a simple calculation shows that*

$$T^{\flat(2)}_{\mathrm{epi}F}(0, (0, 1)) = \mathbb{R} \times \mathbb{R}^2.$$

This means that: (i) $\mathrm{dom}P = \mathbb{R}$ *and* $P(x) = \mathbb{R}^2, \forall x \in \mathrm{dom}P$*; (ii)* $ED_w^{\flat(2)}F(0,(0,1)) = \emptyset$ *for each* $x \in \mathbb{R}$*. Obviously,* $P(x) := \{y \in Y \mid (x,y) \in \mathbb{R} \times \mathbb{R}^2\}$ *does not fulfill the weak domination property for each* $x \in \mathbb{R}$ *and* $\Omega = \emptyset$*. Thus Proposition 4 does not hold here and the condition (iii) of Proposition 4 is essential.*

4. Higher-Order Mond-Weir Type Duality

In this section, by virtue of the higher-order weak adjacent epiderivative of a set-valued map, we establish Mond-Weir duality theorems for a constrained optimization problem under Benson proper efficiency.

Let $E \subseteq X$, $F : E \to 2^Y$ and $G : E \to 2^Z$ be two set-valued maps. We consider the following constrained set-valued optimization problem:

$$\text{(SOP)} \begin{cases} \mathrm{Min}_C & F(x), \\ \;\; s.t. & x \in E, G(x) \cap (-D) \neq \emptyset. \end{cases}$$

Let $M := \{x \in E \mid G(x) \cap (-D) \neq \emptyset\}$ and $F(M) := \cup_{x \in M} F(x)$. We denote $F(x) \times G(x)$ by $(F,G)(x)$. The point $(x_0, y_0) \in E \times Y$ is said to be a feasible solution of (SOP) if $x_0 \in M$ and $y_0 \in F(x_0)$.

Definition 12. *[25] The feasible solution* (x_0, y_0) *is called a Benson proper efficient solution of (SOP) if*

$$\mathrm{clcone}(F(M) + C - \{y_0\}) \cap (-C) = \{0_Y\}.$$

Let $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathrm{gr}(F,G)$, $\nu \in Y^*$, $\omega \in Z^*$ and $x \in \Theta := \mathrm{dom}ED_w^{\flat(m)}(F,G)(\tilde{x}, \tilde{y}, \tilde{z})$. Inspired by [2], We establish a new higher-order Mond-Weir type dual problem (DSOP) of (SOP) as follows:

$$\begin{aligned} \max \quad & \tilde{y} \\ \text{s.t.} \quad & \nu(y) + \omega(z) \geq 0, \forall (y,z) \in ED_w^{\flat(m)}(F,G)(\tilde{x}, \tilde{y}, \tilde{z})(x), x \in \Theta, && (1) \\ & \omega(\tilde{z}) \geqslant 0, && (2) \\ & \nu \in C^{+i}, && (3) \\ & \omega \in D^{+}. && (4) \end{aligned}$$

The point $(\tilde{x}, \tilde{y}, \tilde{z}, \nu, \omega)$ is called a feasible solution of (DSOP) if $(\tilde{x}, \tilde{y}, \tilde{z}, \nu, \omega)$ satisfies conditions (1), (2), (3) and (4) of (DSOP). A feasible solution $(x_0, y_0, z_0, \nu_0, \omega_0)$ is called a maximal solution of (DSOP) if for all $\tilde{y} \in M_D$, $(\{\tilde{y}\} - \{y_0\}) \cap (C \setminus \{0_Y\}) = \emptyset$, where $M_D := \{\tilde{y} \in F(\tilde{x}) \mid (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathrm{gr}(F,G), \nu \in C^{+i}, \omega \in D^+$, and $(\tilde{x}, \tilde{y}, \tilde{z}, \nu, \omega)$ is the feasible solution of (DSOP)$\}$.

Definition 13. *[26] Let* $K \subseteq X$*, the interior tangent cone of* K *at* x_0 *is defined by*

$$IT_K(x_0) := \{\mu \in X \mid \exists \lambda > 0, \forall t \in (0,\lambda), \forall \mu' \in B_X(\mu, \lambda), x_0 + t\mu' \in K\},$$

where $B_X(\mu, \lambda)$ *stands for the closed ball centered at* $\mu \in X$ *and of radius* λ*.*

Theorem 2. *(Weak Duality) Let* E *be a star-shaped set at* $\tilde{x} \in E$ *and* $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathrm{gr}(F,G)$*. Let* (x_0, y_0) *and* $(\tilde{x}, \tilde{y}, \tilde{z}, \nu, \omega)$ *be the feasible solution of (SOP) and (DSOP), respectively. Then the weak duality:* $\nu(y_0) \geq \nu(\tilde{y})$ *holds if the following conditions are satisfied:*

(i) (F,G) *is decreasing-along-rays at* $\tilde{x}$*;*

(ii) (F,G) *is generalized* $C \times D$*-convex at* $\tilde{x}$ *on* E*;*

(iii) the set $P_{(F,G)}(x_0 - \tilde{x}) := \{y \in Y | (x_0 - \tilde{x}, y, z) \in T^{\flat(m)}_{\mathrm{epi}(F,G)}(\tilde{x}, \tilde{y}, \tilde{z})\}$ *fulfills the weak domination property.*

Proof. Since (x_0, y_0) is a feasible solution of (SOP), $G(x_0) \cap (-D) \neq \emptyset$. Take $z_0 \in G(x_0) \cap (-D)$. It follows from (2) and (4) that

$$\omega(z_0 - \tilde{z}) \leqslant 0. \tag{5}$$

From Proposition 4 it follows that $x_0 - \tilde{x} \in S := \text{dom}ED_w^{\flat(m)}(F,G)(\tilde{x},\tilde{y},\tilde{z})$ and

$$(y_0, z_0) - (\tilde{y}, \tilde{z}) \in ED_w^{\flat(m)}(F,G)(\tilde{x},\tilde{y},\tilde{z})(x_0 - \tilde{x}) + C \times D. \tag{6}$$

Noting that $\nu \in C^{+i}$ and $\omega \in D^+$, we have by (1) and (6) that $\nu(y_0 - \tilde{y}) + \omega(z_0 - \tilde{z}) \geqslant 0$. Combining this with (5), one has

$$\nu(y_0 - \tilde{y}) \geqslant 0.$$

Thus $\nu(y_0) \geq \nu(\tilde{y})$ and the proof is complete. □

Theorem 2 is an extension of [2], Theorem 4.1 from cone convexity to generalized cone convexity. Now, we give an example to illustrate that Theorem 2 can apply but [2], Theorem 4.1 dose not.

Example 9. *Let $X = Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$, $F : E \to 2^Y$ be given as $F(x) = \{y \in Y \mid y \geq 0\}$ and $G : E \to 2^Z$ be defined by*

$$G(x) = \begin{cases} \{z \in Z \mid z \geq 0\}, x \leq 0, \\ \qquad \mathbb{R}, \quad x > 0. \end{cases}$$

Then sets of the feasible solutions for (DSOP) and (SOP) are $\{(\tilde{x},\tilde{y},\tilde{z},\nu,\omega) \mid \tilde{x} = 0, \tilde{y} = 0, \tilde{z} \geq 0, \nu \in C^{+i}, \omega = 0\}$ and $\{(x_0, y_0) \mid x_0 \in \mathbb{R}, y_0 \geq 0\}$, respectively. Thus $\nu(y_0) \geq \nu(\tilde{y}) = \nu(0)$ and Theorem 2 holds here. However, [2], Theorem 4.1 is not applicable here because G is not C-convex on E.

Lemma 1. *[27] Let $x_0 \in K \subseteq X$ and $\text{int}K \neq \emptyset$. If K is convex, then*

$$IT_{\text{int}K}(x_0) = \text{intcone}(K - \{x_0\}).$$

The inclusion relation between the generalized second-order adjacent epiderivative and convex cone C and D is established by Wang and Yu in [28], Theorem 5.2. Inspired by [28] , Theorem 5.2, we next introduce the equality of the higher-order weak adjacent epiderivative and convex cone C and D to the proof of the strong duality theory.

Lemma 2. *Let $(x_0, y_0, z_0) \in \text{gr}(F,G)$ and $z_0 \in -D$. If (x_0, y_0) is a Benson proper efficient solution of (SOP), then for all $x \in \Theta := \text{dom}ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)$,*

$$[ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(x) + C \times D + \{(0_Y, z_0)\}] \cap (-((C \setminus \{0_Y\}) \times \text{int}D)) = \emptyset. \tag{7}$$

Proof. We can easily see that (7) is equivalent to

$$[ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(x) + C \times D] \cap (-((C \setminus \{0_Y\}) \times (\text{int}D + \{z_0\}))) = \emptyset. \tag{8}$$

Thus we only need to prove that (8) holds. Suppose on the contrary that there exist $x \in \Theta$, $(y,z) \in ED_w^{\flat(m)}(F,G)$ $(x_0,y_0,z_0)(x)$ and $(c_0, d_0) \in C \times D$ such that

$$z + d_0 \in -(\text{int}D + \{z_0\}) \tag{9}$$

and

$$y + c_0 \in -(C \setminus \{0_Y\}). \tag{10}$$

It follows from $(y,z) \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(x)$ that $(x,y,z) \in T^{\flat(m)}_{\text{epi}(F,G)}(x_0,y_0,z_0)$. Then for any sequence $\{t_n\}$ with $t_n \to 0^+$, there exists $\{(x_n,y_n,z_n)\} \subseteq \text{epi}(F,G)$ such that

$$(\frac{x_n - x_0}{t_n}, \frac{(y_n,z_n)-(y_0,z_0)}{t_n^m}) \to (x,y,z). \tag{11}$$

From (9) and (11), there exists a sufficiently large natural number N_1 such that

$$\begin{aligned}\bar{z}_n := \frac{z_n - z_0 + t_n^m d_0}{t_n^m} &\in -(\text{int}D + \{z_0\}) \subseteq -\text{intcone}(D+\{z_0\}) \\ &\subseteq -IT_{\text{int}D}(-z_0), \forall n > N_1,\end{aligned} \tag{12}$$

where the last inclusion follows from Lemma 1. According to Definition 13, there exists $\lambda > 0$ such that

$$-z_0 + t_n\mu' \in \text{int}D, \forall t_n \in (0,\lambda), \mu' \in B_Y(-\bar{z}_n, \lambda), n > N_1. \tag{13}$$

Since $t_n \to 0^+$, there exists a sufficiently large natural number N_2 with $N_2 \geq N_1$ such that $t_n^m \in (0,\lambda)$. Combining this with (13), one has

$$-z_0 + t_n^m(-\bar{z}_n) \in \text{int}D, \forall n > N_2. \tag{14}$$

From (12) and (14), we have

$$-z_0 - (z_n - z_0 + t_n^m d_0) = -z_n - t_n^m d_0 \in \text{int}D, \forall n > N_2.$$

It follows from $d_0 \in D$, $t_n^m \to 0^+$ and $\text{int}D + D \subseteq \text{int}D$ that

$$z_n \in -\text{int}D, \forall n > N_2. \tag{15}$$

Noting that $\{(x_n,y_n,z_n)\} \subseteq \text{epi}(F,G)$, there exist $x_n \in E$, $\hat{z}_n \in G(x_n)$, $\hat{y}_n \in F(x_n)$ and $(c_n,d_n) \in C \times D$ such that $y_n = \hat{y}_n + c_n$ and $z_n = \hat{z}_n + d_n$. By (15), $\hat{z}_n \in -\text{int}D - \{d_n\} \subseteq -\text{int}D \subseteq -D, \forall n > N_2$. Therefore

$$x_n \in M, \forall n > N_2. \tag{16}$$

Clearly, we have

$$\begin{aligned}\frac{y_n - y_0}{t_n^m} + c_0 = \frac{y_n + t_n^m c_0 - y_0}{t_n^m} &\in \frac{F(x_n) + C - \{y_0\}}{t_n^m} \\ &\subseteq \frac{F(M) + C - \{y_0\}}{t_n^m} \\ &\subseteq \text{clcone}(F(M) + C - \{y_0\}).\end{aligned}$$

It follows from (11) and (16) that $y + c_0 \in \text{clcone}(F(M) + C - \{y_0\})$. Combining this with (10), one has

$$y + c_0 \in \text{clcone}(F(M) + C - \{y_0\}) \cap (-(C \setminus \{0_Y\})),$$

which contradicts that (x_0,y_0) is a Benson proper efficient solution of (SOP). Thus (7) holds and the proof is complete. □

According to Theorem 2.3 of [29], we have the following lemma.

Lemma 3. *[29] Let W be a locally convex space, H and Q be cones in W. If H is closed, Q have a compact base and $H \cap Q = \{0_W\}$, then there is a pointed convex cone $\tilde{A}$ such that $Q \setminus \{0_W\} \subseteq \text{int}\tilde{A}$ and $\tilde{A} \cap H = \{0_W\}$.*

Theorem 3. *(Strong Duality) Let E be a convex subset of X, $(x_0, y_0, z_0) \in \mathrm{gr}(F, G)$ and $z_0 \in -D$. Suppose that the following conditions are satisfied:*

(i) (F, G) is $C \times D$-convex on E;

(ii) $P(x) := \{(y, z) \in Y \times Z \mid (x, y, z) \in T^{\flat(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0)\}$ fulfills the weak domination property for all $x \in \mathrm{dom}P$;

(iii) C has a compact base;

(iv) (x_0, y_0) be a Benson proper efficient solution of (SOP);

(v) for any $x \in E, G(x) \cap (-D) \neq \emptyset$.

Then there exist $\nu \in C^{+i}$ and $\omega \in D^+$ such that $(x_0, y_0, z_0, \nu, \omega)$ is a maximal solution of (DSOP).

Proof. Define
$$\Psi := ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(\Theta) + C \times D + \{(0_Y, z_0)\},$$
where $\Theta := \mathrm{dom}ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)$.

Step 1. We firstly prove that Ψ is a convex set. Indeed, it is sufficient to show the convexity of $\Psi_0 := \Psi - \{(0_Y, z_0)\}$.

Let $(y_i, z_i) \in \Psi_0$ $(i = 1, 2)$. Then there exist $x_i \in \Theta$, $(y'_i, z'_i) \in ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(x_i)$ and $(c_i, d_i) \in C \times D$ $(i = 1, 2)$ such that

$$(y_i, z_i) = (y'_i, z'_i) + (c_i, d_i) \quad (i = 1, 2). \tag{17}$$

According to the definition of $ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)$, one has $(x_i, y'_i, z'_i) \in T^{\flat(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0)$ $(i = 1, 2)$.

Since (F, G) is $C \times D$-convex on E, $\mathrm{epi}(F, G)$ is a convex set. From Proposition 3, $T^{\flat(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0)$ is a convex set. So for any $t \in [0, 1]$,

$$t(x_1, y'_1, z'_1) + (1 - t)(x_2, y'_2, z'_2) \in T^{\flat(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0).$$

By (ii), we have

$$\begin{aligned} t(y'_1, z'_1) + (1 - t)(y'_2, z'_2) &\in ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(tx_1 + (1 - t)x_2) + \mathrm{int}(C \times D) \cup \{(0_Y, 0_Z)\} \\ &\subseteq ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(tx_1 + (1 - t)x_2) + C \times D. \end{aligned}$$

Combining this with (17), one has

$$t(y_1, z_1) + (1 - t)(y_2, z_2) \in \Psi_0 + C \times D = \Psi_0.$$

Therefore Ψ_0 is a convex set and so $\Psi = \Psi_0 + \{(0_Y, z_0)\}$ is a convex set.

Step 2. We prove that there exist $\nu \in C^{+i}$ and $\omega \in D^+$ such that $(x_0, y_0, z_0, \nu, \omega)$ is a feasible solution of (DSOP).

Define
$$\Phi := \mathrm{clcone}\Psi.$$

Since Ψ is a convex set, Φ is a convex cone. According to Lemma 2, we have

$$\Phi \cap (-((C \setminus \{0_Y\}) \times \mathrm{int}D)) = \emptyset. \tag{18}$$

Hence, we can conclude
$$\Phi \cap (-(C \times \{0_Z\})) = \{(0_Y, 0_Z)\}. \tag{19}$$

In fact, assume that (19) does not hold. Since Φ is a cone, there exists $b \in -C \setminus \{0_Y\}$ such that

$$(b, 0_Z) \in \Phi \cap (-((C \setminus \{0_Y\}) \times \{0_Z\})).$$

Then there exist $x^n \in \Theta$, $(y^n, z^n) \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(x^n)$, $(c_n, d_n) \in C \times D$ and $\lambda_n \geq 0$ such that

$$b = \lim_{n\to\infty} \lambda_n (y^n + c_n). \tag{20}$$

According to the definition of $ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)$, for any $t_k \to 0^+$, there exists $(x_k^n, y_k^n, z_k^n) \in \mathrm{epi}(F,G)$ such that

$$\lim_{k\to\infty} \left(\frac{x_k^n - x_0}{t_k}, \frac{y_k^n - y_0 + t_k^m c_n}{t_k^m}, \frac{z_k^n - z_0 + t_k^m d_n}{t_k^m}\right) = (x^n, y^n + c_n, z^n + d_n). \tag{21}$$

This together with condition (v) implies

$$\begin{aligned} \lambda_n \frac{y_k^n - y_0 + t_k^m c_n}{t_k^m} &\in \lambda_n \frac{F(x_k^n) + C - \{y_0\} + t_k^m c_n}{t_k^m} \\ &\subseteq \mathrm{clcone}[F(E) + C - \{y_0\}] \\ &\subseteq \mathrm{clcone}[F(M) + C - \{y_0\}]. \end{aligned} \tag{22}$$

It follows from (20), (21), (22) and $b \in -(C \setminus \{0_Y\})$ that

$$b \in \mathrm{clcone}(F(M) + C - \{y_0\}) \cap (-(C \setminus \{0_Y\})),$$

which contradicts that (x_0, y_0) is a Benson proper efficient solution of (SOP). Thus (19) holds.

Since C has a compact base, $-(C \times \{0_Z\})$ also has a compact base. Combining this with (19) and Lemma 3, replacing H and Q with Φ and $-(C \times \{0_Z\})$, there exists a pointed convex cone $\tilde{A}$ such that

$$-(C \times \{0_Z\}) \backslash (0_Y, 0_Z) \subseteq \mathrm{int}\tilde{A} \tag{23}$$

and

$$\Phi \cap \tilde{A} = \{(0_Y, 0_Z)\}. \tag{24}$$

Let $\tilde{B} := A \cup \{(0_Y, 0_Z)\}$, where $A := -((C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\})) + \tilde{A}$. Thus $\tilde{B}$ is a convex cone.

Next, we further prove that $\tilde{B}$ is a pointed cone. According to Proposition 1, we get $(0_X, 0_Y, 0_Z) \in T_{\mathrm{epi}(F,G)}^{\flat(m)}(x_0, y_0, z_0)$. Combining this with the weak domination property of P, we get

$$(0_Y, 0_Z) \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(0_X) + C \times D. \tag{25}$$

For $z_0 \in G(x_0) \cap (-D)$ and $(c,d) \in C \times D$, we have

$$\begin{aligned} (c,d) &= (0_Y, 0_Z) + (c, d - z_0) + (0_Y, z_0) \\ &\in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(0_X) + C \times D + \{(0_Y, z_0)\} \\ &\subseteq \Phi, \end{aligned}$$

and so

$$C \times D \subseteq \Phi. \tag{26}$$

It follows from (24) and (26) that $(C \times D) \cap \tilde{A} = \{(0_Y, 0_Z)\}$. Hence,

$$((C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\})) \cap \tilde{A} = \emptyset.$$

Combining with the definition of A, one has

$$(0_Y, 0_Z) \notin A. \tag{27}$$

Thus

$$A \cap (-A) = \emptyset. \tag{28}$$

To obtain this result, we suppose on the contrary that there exists $(c, d) \in A \cap (-A)$. Then there exist $(c_i, d_i) \in (C \setminus \{0_Y\}) \times (\text{int}D \cup \{0_Z\})$ $(i = 1, 2)$ and $(c'_i, d'_i) \in \tilde{A}$ $(i = 1, 2)$ such that

$$(c, d) = -(c_1, d_1) + (c'_1, d'_1)$$

and

$$(c, d) = (c_2, d_2) - (c'_2, d'_2).$$

So

$$\begin{aligned}
&(-(c_1, d_1) + (c'_1, d'_1)) - ((c_2, d_2) - (c'_2, d'_2)) \\
&= -(c_1 + c_2, d_1 + d_2) + (c'_1 + c'_2, d'_1 + d'_2) \\
&= (0_Y, 0_Z) \in A,
\end{aligned}$$

which contradicts (27). Therefore (28) holds. Then $\tilde{B}$ is a pointed convex cone and $(0_Y, 0_Z) \notin \text{int}\tilde{B}$.

Now, we can conclude

$$\Phi \cap \tilde{B} = \{(0_Y, 0_Z)\}. \tag{29}$$

To see the conclusion, we suppose on the contrary that there exists $(y, z) \neq (0_Y, 0_Z)$ such that

$$(y, z) \in \Phi \cap \tilde{B}, \tag{30}$$

because $\tilde{B}$ is a pointed convex cone and Φ is a convex cone. From the definition of $\tilde{B}$, there exist $(y_1, z_1) \in -((C \setminus \{0_Y\}) \times (\text{int}D \cup \{0_Z\}))$ and $(y_2, z_2) \in \tilde{A}$ such that

$$(y, z) = (y_1, z_1) + (y_2, z_2).$$

According to the definition of Φ, there exist $x'_n \in \Theta$, $(y'_n, z'_n) \in ED_w^{\flat(m)}(F, G)(x_0, y_0, z_0)(x'_n)$, $(c'_n, d'_n) \in C \times D$ and $\lambda'_n \geq 0$ such that

$$(y, z) = \lim_{n\to\infty} \lambda'_n (y'_n + c'_n, z'_n + d'_n + z_0).$$

Since $(y, z) \neq (0_Y, 0_Z)$, without loss of generality, we may assume that $\lambda'_n > 0$. It follows from the definition of Φ that

$$\begin{aligned}
(y, z) - (y_1, z_1) &= \lim_{n\to\infty} \lambda'_n (y'_n + c'_n, z'_n + d'_n + z_0) - (y_1, z_1) \\
&= \lim_{n\to\infty} \lambda'_n (y'_n + c'_n - \frac{y_1}{\lambda'_n}, z'_n + d'_n - \frac{z_1}{\lambda'_n} + z_0) \\
&\in \Phi,
\end{aligned}$$

and so

$$(y_2, z_2) = (y, z) - (y_1, z_1) \in \Phi \cap \tilde{A} = \{(0_Y, 0_Z)\}.$$

Thus

$$(y, z) = (y_1, z_1) \in -((C \setminus \{0_Y\}) \times \text{int}D). \tag{31}$$

By (30) and (31), we have

$$(y,z) \in \Phi \cap (-((C \setminus \{0_Y\}) \times \mathrm{int}D)),$$

which contradicts (18).

We claim that

$$-((C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\})) \subseteq \mathrm{int}\tilde{B}. \tag{32}$$

To obtain this conclusion, we replace B and C in [30], Theorem 2.2 with $-((C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\}))$ and $\mathrm{int}\tilde{A}$, respectively, which together with the fact: $(0_Y, 0_Z) \notin \mathrm{int}\tilde{B}$ yields that

$$\mathrm{int}\tilde{B} = -(C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\}) + \mathrm{int}\tilde{A}. \tag{33}$$

Let $c \in C \setminus \{0_Y\}$ and $d \in \mathrm{int}D \cup \{0_Z\}$. Then by (23) and (33), one has

$$-(c,d) = -(\frac{c}{2}, d) - (\frac{c}{2}, 0_Z) \in -((C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\})) + \mathrm{int}\tilde{A} = \mathrm{int}\tilde{B},$$

and so (32) holds.

According to the separation theorem for convex set and (29), there exist $\nu \in Y^*$ and $\omega \in Z^*$ such that

$$\nu(\bar{y}) + \omega(\bar{z}) < 0, \forall (\bar{y}, \bar{z}) \in \mathrm{int}\tilde{B} \tag{34}$$

and

$$\nu(\check{y}) + \omega(\check{z}) \geq 0, \forall (\check{y}, \check{z}) \in \Phi. \tag{35}$$

By (32) and (34), we have

$$\nu(\bar{y}) + \omega(\bar{z}) > 0, \forall (\bar{y}, \bar{z}) \in (C \setminus \{0_Y\}) \times (\mathrm{int}D \cup \{0_Z\}). \tag{36}$$

Taking $\bar{z} = 0_Z$ in (36), one has $\nu(\bar{y}) > 0, \forall \bar{y} \in C \setminus \{0_Y\}$, thus $\nu \in C^{+i}$. For any $\varepsilon > 0$, take $\bar{y} \in (C \setminus \{0_Y\}) \cap B(0_Y, \varepsilon)$ in (36). Then we can observe that $\omega(\bar{z}) \geq 0, \forall \bar{z} \in \mathrm{int}D$, which implies $\omega \in D^+$.

It follows from (35) that

$$\nu(y) + \omega(z) \geq 0, \forall (y,z) \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(\Theta) + C \times D + \{(0_Y, z_0)\}. \tag{37}$$

Together with (25), we get $\omega(z_0) \geq 0$. It follows from $z_0 \in -D$ and $\omega \in D^+$ that $\omega(z_0) \leq 0$. Thus,

$$\omega(z_0) = 0.$$

Combining this with (37), one has

$$\nu(y') + \omega(z') \geq 0, \forall (y', z') \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(\Theta) + C \times D,$$

and so

$$\nu(y) + \omega(z) \geq 0, \forall (y,z) \in ED_w^{\flat(m)}(F,G)(x_0,y_0,z_0)(\Theta).$$

Thus $(x_0, y_0, z_0, \nu, \omega)$ is a feasible solution of (DSOP).

Step 3. We prove that $(x_0, y_0, z_0, \nu, \omega)$ is a maximal solution of (DSOP).

Suppose on the contrary that there exists a feasible solution $(\hat{x}, \hat{y}, \hat{z}, \nu', \omega')$ such that $\hat{y} - y_0 \in C \setminus \{0_Y\}$. By $\nu' \in C^{+i}$, we have

$$\nu'(\hat{y}) > \nu'(y_0) \tag{38}$$

Since (x_0, y_0) is a feasible solution of (SOP), it follows from Theorem 2 that $\nu'(y_0) \geqslant \nu'(\hat{y})$, which contradicts (38). The proof is complete. □

Theorem 4. *(Converse Duality) Let E be a star-shaped set at $x_0 \in E$. Let $y_0 \in F(x_0), z_0 \in G(x_0) \cap (-D)$, $\nu \in C^{+i}$ and $\omega \in D^+$ such that $(x_0, y_0, z_0, \nu, \omega)$ is a feasible solution of (DSOP). Then (x_0, y_0) is a Benson proper efficient solution of (SOP) if the following conditions are satisfied:*

(i) (F, G) is decreasing-along-rays at x_0;

(ii) (F, G) is a generalized $C \times D$-convex at x_0 on E;

(iii) the set $P_{(F,G)}(x - x_0) := \{y \in Y \mid (x - x_0, y, z) \in T^{\flat(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0)\}$ fulfills the weak domination property for all $x \in \mathrm{dom}P_{(F,G)}$.

Proof. It follows from (1), (3) and (4) that

$$\begin{aligned} \nu(y) + \omega(z) \geq 0, \forall (y, z) \in ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(x) + C \times D, \\ \forall x \in \Theta := \mathrm{dom}ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0). \end{aligned} \tag{39}$$

According to Proposition 4, we get

$$\begin{aligned} (y - y_0, z - z_0) \in ED^{\flat(m)}_w(F, G)(x_0, y_0, z_0)(x - x_0) + C \times D, \\ \forall x \in M, y \in F(x), z \in G(x) \cap (-D). \end{aligned} \tag{40}$$

By (2), we have $\omega(z_0) \geq 0$. It follows from $z_0 \in G(x_0) \cap (-D)$ and $\omega \in D^+$ that $\omega(z_0) \leqslant 0$, thus $\omega(z_0) = 0$. Then

$$\omega(z - z_0) = \omega(z) - \omega(z_0) = \omega(z) \leqslant 0, \forall z \in G(x) \cap (-D), x \in M. \tag{41}$$

It follows from (39), (40) and (41) that

$$\nu(y - y_0) \geq 0, \forall y \in F(x), x \in M.$$

Further more, we can get

$$\nu(y + c - y_0) \geq 0, \forall y \in F(x), x \in M,$$

and so

$$\nu(y) \geq 0, \forall y \in \mathrm{clcone}(F(M) + C - \{y_0\}). \tag{42}$$

Assume that the feasible solution (x_0, y_0) is not a Benson proper efficient solution of (SOP). Then there exists $y' \in -(C \setminus \{0_Y\})$ such that $y' \in \mathrm{clcone}(F(M) + C - \{y_0\})$. This together with (42) implies that

$$\nu(y') \geq 0. \tag{43}$$

It follows from $\nu \in C^{+i}$ and $y' \in -(C \setminus \{0_Y\})$ that $\nu(y') < 0$, which contradicts (43). Thus (x_0, y_0) is a Benson proper efficient of (SOP) and the proof is complete. □

Remark 4. *Example 9 also illustrates that Theorem 4 extends [2], Theorem 4.3 from the cone convexity to generalized cone convexity. Indeed, take $(x_0, y_0, z_0) = (0, 0, 0)$. Then simple calculations show that*

$$\begin{aligned} T^{\flat(2)}_{\mathrm{epi}(F,G)}(0, 0, 0) = & \{(x, y, z) \in X \times Y \times Z \mid x \leq 0, y \geq 0, z \geq 0\} \cup \\ & \{(x, y, z) \in X \times Y \times Z \mid x > 0, y \geq 0, z \in \mathbb{R}\} \end{aligned}$$

and

$$ED^{\flat(2)}_w(F, G)(0, 0, 0)(x) = \begin{cases} \{(y, z) \in Y \times Z \mid y = 0, z \geq 0\} \cup \{(y, z) \in Y \times Z \mid y \geq 0, z = 0\}, x \leq 0, \\ \{(y, z) \in Y \times Z \mid y = 0, z \in \mathbb{R}\}, x > 0. \end{cases}$$

Then we can choose $\nu = 1$ *and* $\omega = 0$ *such that* $(x_0, y_0, z_0, \nu, \omega) = (0, 0, 0, 1, 0)$ *is a feasible solution of (DSOP). It is easy to show that the all conditions of Theorem 4 are fulfilled and* $(0, 0)$ *is a Benson proper efficient solution of (SOP). Thus Theorem 4 holds here. However, [2], Theorem 4.3 is not applicable here because G is not C-convex on E.*

Author Contributions: The authors made equal contributions to this paper.

Funding: This research was funded by Chongqing Jiaotong University Graduate Education Innovation Foundation Project (No. 2018S0152), Chongqing Natural Science Foundation Project of CQ CSTC(Nos. 2015jcyjA30009, 2015jcyjBX0131, 2017jcyjAX0382), the Program of Chongqing Innovation Team Project in University (No. CXTDX201601022) and the National Natural Science Foundation of China (No. 11571055). Ching-Feng Wen was supported by the Taiwan MOST [grant number 107-2115-M-037-001].

Conflicts of Interest: The author declares no conflict of interest.

References

1. Corley, H.W. Optimality conditions for maximizations of set-valued functions. *J. Optim. Theory Appl.* **1988**, *58*, 1–10. [CrossRef]
2. Chen, C.R.; Li, S.J.; Teo, K.L. Higher order weak epiderivatives and applications to duality and optimality conditions. *Comput. Math. Appl.* **2009**, *57*, 1389–1399. [CrossRef]
3. Li, S.J.; Teo, K.L.; Yang, X.Q. Higher order Mond-Weir duality for set-valued optimization. *J. Comput. Appl. Math.* **2008**, *217*, 339–349. [CrossRef]
4. Wang, Q.L.; Li, S.J.; Chen, C.R. Higher-order generalized adjacent derivative and applications to duality for set-valued optimization. *Taiwan. J. Math.* **2011**, *15*, 1021–1036. [CrossRef]
5. Wen, C.F.; Wu, H.C. Approximate solutions and duality theorems for continuous-time linear fractional problems. *Numer. Funct. Anal. Optim.* **2012**, *33*, 80–129. [CrossRef]
6. Anh, N.L.H. On higher-order mixed duality in set-valued optimization. *Bull. Malays. Math. Sci. Soc.* **2018**, *41*, 723–739. [CrossRef]
7. Li, S.J.; Sun, X.K.; Zhu, S.K. Higher-order optimality conditions for sirict minimality in set-valued optimization. *J. Nonlinear Convex Anal.* **2012**, *13*, 281–291.
8. Anh, N.L.H. Mixed type duality for set-valued optimization problems via higher-order radial epiderivatives. *Numer. Func. Anal. Optim.* **2016**, *37*, 823–838. [CrossRef]
9. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Birkhauser: Boston, MA, USA, 1990.
10. Luc, D.T. *Theory of Vector Optimization*; Springer: Berlin, Germany, 1989.
11. Jahn, J.; Rauh, R. Contingent epiderivatives and set-valued optimization. *Math. Meth. Oper. Res.* **1997**, *46*, 193–211. [CrossRef]
12. Chen, G.Y.; Jahn, J. Optimality conditions for set-valued optimization problems. *Math. Methods Oper. Res.* **1998**, *48*, 187–200. [CrossRef]
13. Jahn, J.; Khan, A.A.; Zeilinger, P. Second-order optimality conditions in set optimization. *J. Optim. Theory Appl.* **2005**, *125*, 331–347. [CrossRef]
14. Li, S.J.; Chen, C.R. Higher order optimality conditions for Henig efficient solutions in set-valued optimization. *J. Math. Anal. Appl.* **2006**, *323*, 1184–1200. [CrossRef]
15. Wang, Q.L.; Li, S.J.; Teo, K.L. Higher order optimality conditions for weakly efficient solutions in nonconvex set-valued optimization. *Optim. Lett.* **2010**, *4*, 425–437. [CrossRef]
16. Anh, N.L.H.; Khanh, P.Q. Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives. *J. Glob. Optim.* **2013**, *56*, 519–536. [CrossRef]
17. Studniarski, M. Necessary and sufficient conditions for isolated local minima of nonsmooth functions. *SIAM J. Control Optim.* **1986**, *24*, 1044–1049. [CrossRef]
18. Anh, N.L.H. Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality. *Positivity* **2014**, *18*, 449–473. [CrossRef]
19. Anh, N.L.H. Higher-order optimality conditions for strict and weak efficient solutions in set-valued optimization. *Positivity* **2016**, *20*, 499–514. [CrossRef]
20. Anh, N.L.H. Higher-order generalized Studniarski epiderivative and its applications in set-valued optimization. *Positivity* **2018**, *22*, 1371–1385. [CrossRef]

21. Lalitha, C.S.; Arora, R. Weak Clarke epiderivative in set-valued optimization. *J. Math. Anal. Appl.* **2008**, *342*, 704–714. [CrossRef]
22. Jahn, J. *Vector Optimization: Theory, Applications, and Extensions*; Springer: Berlin, Germany, 2004.
23. Bobylev, N.A. The Helly theorem for star-shaped sets. *J. Math. Sci.* **2001**, *105*, 1819–1825. [CrossRef]
24. Chinaie, M.; Zafarani, J. Image Space Analysis and Scalarization of Multivalued optimization. *J. Optim. Theory Appl.* **2009**, *142*, 451–467. [CrossRef]
25. Long, X.J.; Peng, J.W.; Wong, M.M. Generalized radial epiderivatives and nonconvex set-valued optimization problems. *Appl. Anal.* **2012**, *91*, 1891–1900. [CrossRef]
26. Jimenez, B.; Novo, V. Second order necessary conitions in set cnstrained differentiable vector optimization. *Math. Methods Oper. Res.* **2013**, *58*, 299–317. [CrossRef]
27. Gong, X.H.; Dong, H.B.; Wang, S.Y. Optimality conditions for proper efficient solutions of vector set-valued optimization. *J. Math. Anal. Appl.* **2003**, *284*, 332–350. [CrossRef]
28. Wang, Q.L.; Yu, G.L. Second-order optimality conditions for set-valued optimization problems under benson proper efficiency. *Abstr. Appl. Anal.* **2011**. [CrossRef]
29. Dauer, J.P.; Saleh, O.A. A characterization of proper minimal points as a solution of sunlinear optimization problems. *J. Math. Anal. Appl.* **1993**, *178*, 227–246. [CrossRef]
30. Zowe, J. A remark on a regularity assumption for the mathematical propramming pronlem in Banach spaces. *J. Optim. Theoy Appl.* **1978**, *25*, 375–381. [CrossRef]