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Nadler and Kannan Type Set Valued Mappings in M-Metric Spaces and an Application

Pradip R. Patle [1], Deepesh Kumar Patel [1], Hassen Aydi [2,3,*], Dhananjay Gopal [4] and Nabil Mlaiki [5]

[1] Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India; pradip.patle12@gmail.com (P.R.P.); deepesh456@gmail.com (D.K.P.)
[2] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia
[3] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[4] Department of Applied Mathematics & Humanities, S.V. National Institute of Technology, Surat 395007, Gujarat, India; gopaldhananjay@yahoo.in
[5] Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia; nmlaiki@psu.edu.sa or nmlaiki2012@gmail.com
* Correspondence: hassen.aydi@isima.rnu.tn

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Abstract: This article intends to initiate the study of Pompeiu–Hausdorff distance induced by an M-metric. The Nadler and Kannan type fixed point theorems for set-valued mappings are also established in the said spaces. Moreover, the discussion is supported with the aid of competent examples and a result on homotopy. This approach improves the current state of art in fixed point theory.

Keywords: homotopy; M-metric; M-Pompeiu–Hausdorff type metric; multivalued mapping; fixed point

MSC: Primary 47H10; Secondary 54H25, 05C40

1. Introduction

With the introduction of Banach's contraction principle (BCP), the fixed point theory advanced in various directions. Nadler [1] obtained the fundamental fixed point result for set-valued mappings using the notion of Pompeiu–Hausdorff metric which is an extension of the BCP. Later on, many fixed point theorists followed the findings of Nadler and contributed significantly to the development of theory (cf. S. Reich [2,3]).

On the other hand, in order to investigate the semantics of data flow networks; Matthews [4] coined the concept called as partial metric spaces which are used efficiently while building models in computation theory. On the inclusion of partial metric spaces into literature, many fixed point theorems were established in this setting, see [5–16]. Recently, Asadi et al. [17] brought the notion of an M-metric as a real generalization of a partial metric into the literature. They also obtained the M-metric version of the fixed point results of Banach and Kannan. Also, some fixed point theorems have been established in M-metric spaces endowed with a graph, see [18].

In this work, we introduce the M-Pompeiu–Hausdorff type metric. Furthermore, we extend the fixed point theorems of Nadler and Kannan to M-metric spaces for set-valued mappings. Finally, homotopy results for M-metric spaces are discussed.

2. Preliminaries

The symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ represent respectively set of all natural numbers, real numbers and nonnegative real numbers. Let us recall some of the concepts for simplicity in understanding.

Definition 1 ([4]). *Let X be a nonempty set. Then a partial metric is a function $p : X \times X \to \mathbb{R}^+$ satisfying following conditions:*

(p_1) $a = b \Longleftrightarrow p(a,a) = p(a,b) = p(b,b)$;
(p_2) $p(a,a) \leq p(a,b)$;
(p_3) $p(a,b) = p(b,a)$;
(p_4) $p(a,b) \leq p(a,c) + p(c,b) - p(c,c)$;

for all $a, b, c \in X$. The pair (X, p) is called a partial metric space.

The concept of an M-metric [17] defined in following definition extends and generalize the notion of partial metric.

Definition 2 ([17]). *Let X be a non empty set. Then an M-metric is a function $m : X \times X \to \mathbb{R}^+$ satisfying the following conditions:*

(m_1) $m(a,a) = m(b,b) = m(a,b) \Leftrightarrow a = b$;
(m_2) $m_{ab} \leq m(a,b)$ where $m_{ab} := \min\{m(a,a), m(b,b)\}$;
(m_3) $m(a,b) = m(b,a)$;
(m_4) $(m(a,b) - m_{ab}) \leq (m(a,c) - m_{ac}) + (m(c,b) - m_{cb})$;

for all $a, b, c \in X$. The pair (X, m) is called an M-metric space.

Remark 1 ([17]). *Let us denote $M_{ab} := \max\{m(a,a), m(b,b)\}$, where m is an M-metric on X. Then for every $a, b \in X$, we have*

(1) $0 \leq M_{ab} + m_{ab} = m(a,a) + m(b,b)$,
(2) $0 \leq M_{ab} - m_{ab} = |m(a,a) - m(b,b)|$,
(3) $M_{ab} - m_{ab} \leq (M_{ac} - m_{ac}) + (M_{cb} - m_{cb})$.

Example 1 ([17]). *Let m be an M-metric on X. Then*

(1) $m^w(a,b) = m(a,b) - 2m_{ab} + M_{ab}$,

(2) $$m^s(a,b) = \begin{cases} m(a,b) - m_{ab} & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases}$$

are ordinary metrics on X.

Two new examples of M-metrics are as follows:

Example 2. *Let $X = [0, \infty)$. Then*

(a) $m_1(a,b) = |a - b| + \frac{a+b}{2}$,
(b) $m_2(a,b) = |a - b| + \frac{a+b}{3}$

are M-metrics on X.

Let $B_m(a, \eta) = \{b \in X : m(a,b) < m_{ab} + \eta\}$ be the open ball with center a and radius $\eta > 0$ in M-metric space (X, m). The collection $\{B_m(a, \eta) : a \in X, \eta > 0\}$, acts as a basis for the topology τ_m (say) on M-metric X.

Remark 2 ([17]). *τ_m is T_0 but not Hausdorff.*

Definition 3 ([17]). *Let $\{a_k\}$ be a sequence in M-metric spaces (X, m).*

(1) *$\{a_k\}$ is called M-convergent to $a \in X$ if and only if*

$$\lim_{k\to\infty}(m(a_k, a) - m_{a_k a}) = 0.$$

(2) *If $\lim_{k,j\to\infty}(m(a_k, a_j) - m_{a_k a_j})$ and $\lim_{k,j\to\infty}(M_{a_k a_j} - m_{a_k a_j})$ exist and finite then the sequence $\{a_k\}$ is called M-Cauchy.*

(3) *If every M-Cauchy sequence $\{a_k\}$ is M-convergent, with respect to τ_m, to $a \in X$ such that $\lim_{k\to\infty}(m(a_k, a) - m_{a_k a}) = 0$ and $\lim_{k\to\infty}(M_{a_k a} - m_{a_k a}) = 0$ then (X, m) is called M-complete.*

Lemma 1 ([17]). *Let $\{a_k\}$ be a sequence in M-metric spaces (X, m). Then*

(i) *$\{a_k\}$ is M-Cauchy if and only if it is a Cauchy sequence in the metric space (X, m^w).*
(ii) *(X, m) is M-complete if and only if (X, m^w) is complete.*

Example 3. *Let X and $m_1, m_2 : X \times X \to [0, \infty)$ be as defined in Example 2 for all $a, b \in X$. Then (X, m_1) and (X, m_2) are M-complete. Indeed, $(X, m^w) = ([0, \infty), k|x - y|)$ is a complete metric space, where $k = \frac{5}{2}$ for m_1 and $k = 2$ for m_2.*

Lemma 2 ([17]). *Let $a_k \to a$ and $b_k \to b$ as $k \to \infty$ in (X, m). Then as $k \to \infty$, $(m(a_k, b_k) - m_{a_k b_k}) \to (m(a, b) - m_{ab})$.*

Lemma 3 ([17]). *Let $a_k \to a$ as $k \to \infty$ in (X, m). Then $(m(a_k, b) - m_{a_k b}) \to (m(a, b) - m_{ab})$, $k \to \infty$, for all $b \in X$.*

Lemma 4 ([17]). *Let $a_k \to a$ and $a_k \to b$ as $k \to \infty$ in (X, m). Then $m(a, b) = m_{ab}$. Further, if $m(a, a) = m(b, b)$, then $a = b$.*

Lemma 5 ([17]). *Let $\{a_k\}$ be a sequence in (X, m) such that for some $r \in [0, 1)$, $m(a_{k+1}, a_k) \le r m(a_k, a_{k-1})$, $k \in \mathbb{N}$ then*

(a) $\lim_{k\to\infty} m(a_k, a_{k-1}) = 0$;
(b) $\lim_{k\to\infty} m(a_k, a_k) = 0$;
(c) $\lim_{k,j\to\infty} m_{a_k, a_j} = 0$;
(d) *$\{a_k\}$ is M-Cauchy.*

3. *M*-Pompeiu–Hausdorff Type Metric

The concept of a partial Hausdorff metric is defined in [19,20]. Following them we initiate the notion of an M-Pompeiu–Hausdorff type metric induced by an M-metric in this section. Let us begin with the following definition.

Definition 4. *A subset A of an M-metric space (X, m) is called bounded if for all $a \in A$, there exist $b \in X$ and $K \ge 0$ such that $a \in B_m(b, K)$, that is, $m(a, b) < m_{ba} + K$.*

Let $\mathcal{CB}^m(X)$ denotes the family of all nonempty, bounded, and closed subsets in (X, m). For $P, Q \in \mathcal{CB}^m(X)$, define

$$\mathcal{H}_m(P, Q) = \max\{\delta_m(P, Q), \delta_m(Q, P)\},$$

where $\delta_m(P, Q) = \sup\{m(a, Q) : a \in P\}$ and $m(a, Q) = \inf\{m(a, b) : b \in Q\}$.

Let $\overline{P}$ denote the closure of P with respect to M-metric m. Note that P is closed in (X,m) if and only if $\overline{P} = P$.

Lemma 6. *Let P be any nonempty set in an M-metric space (X,m), then $a \in \overline{P}$ if and only if $m(a,P) = \sup_{x\in P} m_{ax}$.*

Proof.

$$\begin{aligned}
a \in \overline{P} &\Leftrightarrow B_m(a,\eta) \cap P \neq \emptyset, \text{ for all } \eta > 0\\
&\Leftrightarrow m(a,x) < m_{ax} + \eta, \text{ for some } x \in P\\
&\Leftrightarrow m(a,x) - m_{ax} < \eta\\
&\Leftrightarrow \inf\{m(a,x) - m_{ax} : x \in P\} = 0\\
&\Leftrightarrow \inf\{m(a,x) : x \in P\} = \sup\{m_{ax} : x \in P\}\\
&\Leftrightarrow m(a,P) = \sup_{x\in P} m_{ax}.
\end{aligned}$$

□

Proposition 1. *Let $P, Q, R \in \mathcal{CB}^m(X)$, then we have*

(a) $\delta_m(P,P) = \sup_{a\in P}\{\sup_{b\in P} m_{ab}\}$;
(b) $(\delta_m(P,Q) - \sup_{a\in P}\sup_{b\in Q} m_{ab}) \leq (\delta_m(P,R) - \inf_{a\in P}\inf_{c\in R} m_{ac}) + (\delta_m(R,Q) - \inf_{c\in R}\inf_{b\in Q} m_{cb})$.

Proof.

(a) Since $P \in \mathcal{CB}^m(X)$, $P = \overline{P}$. Then from Lemma 6, $m(a,P) = \sup_{x\in P} m_{ax}$. Therefore, $\delta_m(P,P) = \sup_{a\in P}\{m(a,P)\} = \sup_{a\in P}\{\sup_{x\in P} m_{ax}\}$.
(b) For any $a \in P$, $b \in Q$ and $c \in R$, we have

$$m(a,b) - m_{ab} \leq m(a,c) - m_{ac} + m(c,b) - m_{cb}.$$

We rewrite it as

$$m(a,b) - m_{ab} + m_{ac} + m_{cb} \leq m(a,c) + m(c,b).$$

Since b is arbitrary element in Q, we have

$$m(a,Q) - \sup_{b\in Q} m_{ab} + m_{ac} + \inf_{b\in Q} m_{cb} \leq m(a,c) + m(c,Q).$$

Since $m(c,Q) \leq \delta_m(R,Q)$, we can write above inequality as

$$m(a,Q) - \sup_{b\in Q} m_{ab} + m_{ac} + \inf_{b\in Q} m_{cb} \leq m(a,c) + \delta_m(R,Q).$$

As c is arbitrary in R, we have

$$m(a,Q) - \sup_{b\in Q} m_{ab} + \inf_{c\in R} m_{ac} + \inf_{c\in R}\inf_{b\in Q} m_{cb} \leq m(a,R) + \delta_m(R,Q).$$

We rewrite the above inequality as

$$m(a,Q) + \inf_{c\in R}\inf_{b\in Q} m_{cb} \leq m(a,R) + \delta_m(R,Q) + \sup_{b\in Q} m_{ab} - \inf_{c\in R} m_{ac}.$$

Again, as a is arbitrary in P, we get

$$\delta_m(P,Q) + \inf_{c\in R}\inf_{b\in Q} m_{cb} \le \delta_m(P,R) + \delta_m(R,Q) + \sup_{a\in P}\sup_{b\in Q} m_{ab} - \inf_{a\in P}\inf_{c\in R} m_{ac}.$$

□

Proposition 2. *For any $P, Q, R \in \mathcal{CB}^m(X)$ following are true*

(i) $\mathcal{H}_m(P,P) = \delta_m(P,P) = \sup_{a\in P}\{\sup_{b\in P} m_{ab}\}$;

(ii) $\mathcal{H}_m(P,Q) = \mathcal{H}_m(Q,P)$;

(iii) $\mathcal{H}_m(P,Q) - \sup_{a\in P}\sup_{b\in Q} m_{ab} \le \mathcal{H}_m(P,R) + \mathcal{H}_m(Q,R) - \inf_{a\in P}\inf_{c\in R} m_{ac} - \inf_{c\in R}\inf_{b\in Q} m_{cb}$.

Proof.

(i) From (a) of Proposition 1, we write $\mathcal{H}_m(P,P) = \delta_m(P,P) = \sup_{a\in P}\{\sup_{b\in P} m_{ab}\}$.

(ii) It follows from (m_2) of Definition 2.

(iii) Using (b) of Proposition 1, we have

$$\begin{aligned}
&\mathcal{H}_m(P,Q) = \max\{\delta_m(P,Q), \delta_m(Q,P)\} \\
&\le \max\Big\{[\delta_m(P,R) - \inf_{a\in P}\inf_{c\in R} m_{ac} + \delta_m(R,Q) - \inf_{c\in R}\inf_{b\in Q} m_{cb} + \sup_{a\in P}\sup_{b\in Q} m_{ab}], \\
&\qquad [\delta_m(Q,R) - \inf_{a\in P}\inf_{c\in R} m_{ac} + \delta_m(R,P) - \inf_{c\in R}\inf_{b\in Q} m_{cb} + \sup_{a\in P}\sup_{b\in Q} m_{ab}]\Big\} \\
&\le \max\{\delta_m(P,R), \delta_m(R,P)\} + \max\{\delta_m(Q,R), \delta_m(R,Q)\} \\
&\qquad - \inf_{a\in P}\inf_{c\in R} m_{ac} - \inf_{c\in R}\inf_{b\in Q} m_{cb} + \sup_{a\in P}\sup_{b\in Q} m_{ab} \\
&\le \mathcal{H}_m(P,R) + \mathcal{H}_m(R,Q) - \inf_{a\in P}\inf_{c\in R} m_{ac} - \inf_{c\in R}\inf_{b\in Q} m_{cb} + \sup_{a\in P}\sup_{b\in Q} m_{ab}.
\end{aligned}$$

□

Remark 3. *In general, $\mathcal{H}_m(A,A) \ne 0$ for $A \in \mathcal{CB}^m(X)$. It can be verified through the following example.*

Example 4. *Let $X = [0,\infty)$ and $m(a,b) = \frac{a+b}{2}$, then clearly (X,m) is an M-metric space. In view of (a) of Proposition 1, we have*

$$\mathcal{H}_m([1,2],[1,2]) = \delta_m([1,2],[1,2]) = \sup_{p\in[1,2]}\sup_{q\in[1,2]} m_{pq} = \sup_{p\in[1,2]}\sup_{q\in[1,2]} \min\{p,q\} \ne 0.$$

In view of Proposition 2, we call $\mathcal{H}_m : \mathcal{CB}^m(X) \times \mathcal{CB}^m(X) \to [0,+\infty)$ an M-Pompeiu–Hausdorff type metric induced by m.

Lemma 7. *Let $P, Q \in \mathcal{CB}^m(X)$ and $q > 1$. Then for every $a \in P$, there is at least one $b \in Q$ such that $m(a,b) \le q\mathcal{H}_m(P,Q)$.*

Proof. Assume that there exists an $a \in P$ such that $m(a,b) > q\mathcal{H}_m(P,Q)$ for all $b \in Q$. This implies that

$$\inf_{b\in Q}\{m(a,b)\} \ge q\mathcal{H}_m(P,Q),$$

that is,

$$m(a,Q) \ge q\mathcal{H}_m(P,Q).$$

Note that

$$\mathcal{H}_m(P,Q) \geq \delta_m(P,Q) = \sup_{x\in P} m(x,Q) \geq m(a,Q) \geq q\mathcal{H}_m(P,Q).$$

Since $\mathcal{H}_m(P,Q) \neq 0$, $q \leq 1$, which is a contradiction. □

Lemma 8. Let $P,Q \in \mathcal{CB}^m(X)$ and $r > 0$. For any $a \in P$, there is at least one $b \in Q$ such that $m(a,b) \leq \mathcal{H}_m(P,Q) + r$.

Proof. Assume that there exists $a \in P$ such that $m(a,b) > \mathcal{H}_m(P,Q) + r$ for all $b \in Q$. This implies that

$$\inf_{b\in Q}\{m(a,b)\} \geq \mathcal{H}_m(P,Q) + r,$$

that is,

$$m(a,Q) \geq \mathcal{H}_m(P,Q) + r.$$

Now,

$$\mathcal{H}_m(P,Q) + r \leq m(a,Q) \leq \delta_m(P,Q) \leq \mathcal{H}_m(P,Q).$$

Thus, $r \leq 0$, which is a contradiction. □

4. Fixed Point Results

First, we state the Nadler fixed point theorem in the class of M-metric spaces.

Theorem 1. *Let M-metric space (X,m) be M-complete and $F : X \to \mathcal{CB}^m(X)$ be a multivalued mapping. Suppose there exists $\lambda \in (0,1)$ such that*

$$\mathcal{H}_m(Fa,Fb) \leq \lambda m(a,b), \tag{1}$$

for all $a,b \in X$. Then F admits a fixed point.

Proof. Choose $q = \frac{1}{\sqrt{\lambda}}$ and $r = \sqrt{\lambda}$. Clearly, $q > 1$ and $r < 1$. Let $a_0 \in X$ be arbitrary and $a_1 \in Fa_0$. From Lemma 7, for $q = \frac{1}{\sqrt{\lambda}}$, there exists $a_2 \in Fa_1$ such that

$$m(a_1,a_2) \leq \frac{1}{\sqrt{\lambda}}\mathcal{H}_m(Fa_0,Fa_1). \tag{2}$$

As $\mathcal{H}_m(Fa_0,Fa_1) \leq \lambda m(a_0,a_1)$, so from (2) we have

$$m(a_1,a_2) \leq \frac{1}{\sqrt{\lambda}}\lambda m(a_0,a_1) = \sqrt{\lambda}m(a_0,a_1) = rm(a_0,a_1).$$

Now, from Lemma 7, there exists $a_3 \in Fa_2$ such that

$$m(a_2,a_3) \leq rm(a_1,a_2).$$

Continuing in this way, we get a sequence $\{a_k\}$ of points in X such that $a_{k+1} \in Fa_k$ and for $k \geq 1$,

$$m(a_k,a_{k+1}) \leq rm(a_{k-1},a_k), \tag{3}$$

that is,

$$m(a_k,a_{k+1}) \leq r^k m(a_0,a_1). \tag{4}$$

By Lemma 5, we have

$$\lim_{k\to\infty} m(a_k,a_{k+1}) = 0, \tag{5}$$

$$\lim_{k\to\infty} m(a_k, a_k) = 0, \tag{6}$$

and

$$\lim_{k,j\to\infty} m(a_k, a_j) = 0. \tag{7}$$

Also the sequence $\{a_k\}$ is M-Cauchy. Thus, M-completeness of X yields existence of $a \in X$ such that

$$\lim_{k\to\infty} (m(a_k, a) - m_{a_k a}) = 0.$$

Since $\lim_{k\to\infty} m(a_k, a_k) = 0$, we have

$$\lim_{k\to\infty} m(a_k, a) = 0. \tag{8}$$

From (1) and (8), we have

$$\lim_{k\to\infty} \mathcal{H}_m(Fa_k, Fa) = 0. \tag{9}$$

Now, since $a_{k+1} \in Fa_k$, $m(a_{k+1}, Fa) \leq \mathcal{H}_m(Fa_k, Fa)$. Taking limit as $k \to \infty$ and using (8), we get

$$\lim_{k\to\infty} m(a_{k+1}, Fa) = 0. \tag{10}$$

As $m_{a_{k+1}Fa} \leq m(a_{k+1}, Fa)$, so we have

$$\lim_{k\to\infty} m_{a_{k+1}Fa} = 0. \tag{11}$$

Using (m_4), we have

$$\begin{aligned} m(a, Fa) - \sup_{b\in Fa} m_{ab} &\leq m(a, Fa) - m_{aFa} \\ &\leq m(a, a_{k+1}) - m_{aa_{k+1}} + m(a_{k+1}, Fa) - m_{a_{k+1}Fa}. \end{aligned}$$

Varying limit as $k \to \infty$ and using (8)–(11), we get

$$m(a, Fa) \leq \sup_{b\in Fa} m_{ab}. \tag{12}$$

Since $m_{ab} \leq m(a, b)$ for every $b \in Fa$, this implies that

$$m_{ab} - m(a, b) \leq 0.$$

Thus

$$\sup\{m_{ab} - m(a, b) : b \in Fa\} \leq 0,$$

that is,

$$\sup_{b\in Fa} m_{ab} - \inf_{b\in Fa} m(a, b) \leq 0.$$

This gives

$$\sup_{b\in Fa} m_{ab} \leq m(a, Fa). \tag{13}$$

From (12) and (13), we have

$$m(a, Fa) = \sup_{b\in Fa} m_{ab}.$$

Thus, by Lemma 6, $a \in \overline{Fa} = Fa$. $\square$

Example 5. Let $X = [0,2]$ be endowed with m-metric $m(a,b) = |a-b| + \frac{a+b}{2}$. Then (X,m) is an M-complete M-metric space (as in Example 3). Let $F : X \to \mathcal{CB}^m(X)$ be a mapping defined as

$$F(a) = \left[0, \frac{1}{7}a^2\right] \quad \text{for all} \quad a \in X.$$

We shall show that for $\lambda \in (0,1)$, $\mathcal{H}_m(Fa, Fb) \leq \lambda m(a,b)$, i.e., (1) holds for all $a, b \in X$. We have following three possible cases:

Case I: $a = b = p$. Then $Fa = [0, \frac{1}{7}p^2] = Fb$. Here, for $\lambda \geq \frac{2}{7}$,

$$\mathcal{H}_m(Fa, Fb) = \frac{1}{7}p^2 \leq \lambda p = \lambda m(p,p) = \lambda m(a,b).$$

Case II: $a < b$. Then $Fa = [0, \frac{1}{7}a^2]$, $Fb = [0, \frac{1}{7}b^2]$ and $Fa \subseteq Fb$. In this case,

$$\mathcal{H}_m(Fa, Fb) = \max\left\{\frac{1}{7}a^2, |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2 + \frac{1}{7}b^2}{2}\right\}.$$

Since $a < b$, $\frac{1}{7}a^2 < |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2+\frac{1}{7}b^2}{2}$. So we get

$$\mathcal{H}_m(Fa, Fb) = |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2 + \frac{1}{7}b^2}{2}$$

and $m(a,b) = |a-b| + \frac{(a+b)}{2}$. Then one can see that

$$\begin{aligned}
\mathcal{H}_m(Fa, Fb) &= |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2 + \frac{1}{7}b^2}{2} \\
&= \frac{1}{7}|(a-b)(a+b)| + \frac{1}{7}\frac{a^2+b^2}{2} \\
&= \frac{1}{7}\left[|a-b|(a+b) + \frac{(a+b)^2 - 2ab}{2}\right] \\
&\leq \frac{1}{7}\left[|a-b| + \frac{(a+b)}{2}\right](a+b) \\
&= \frac{(a+b)}{7} m(a,b).
\end{aligned}$$

Case III: $a > b$. Then $Fa = [0, \frac{1}{7}a^2]$, $Fb = [0, \frac{1}{7}b^2]$ and $Fb \subseteq Fa$. In this case,

$$\mathcal{H}_m(Fa, Fb) = \max\left\{\frac{1}{7}b^2, |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2 + \frac{1}{7}b^2}{2}\right\}.$$

Since $b < a$, $\frac{1}{7}b^2 < |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2+\frac{1}{7}b^2}{2}$. So, we get

$$\mathcal{H}_m(Fa, Fb) = |\frac{1}{7}a^2 - \frac{1}{7}b^2| + \frac{\frac{1}{7}a^2 + \frac{1}{7}b^2}{2}$$

and $m(a,b) = |a-b| + \frac{(a+b)}{2}$. Following Case II, one can easily show that

$$\mathcal{H}_m(Fa, Fb) \leq \frac{(a+b)}{7} m(a,b).$$

From above three cases, it is clear that (1) is satisfied for $\lambda \geq \frac{4}{7}$. Thus, all the required conditions of Theorem 1 are satisfied. Hence F admits a fixed point, which is $a = 0$.

Next, we present our fixed point result corresponding to multivalued Kannan contractions in M-metric spaces.

Theorem 2. *Let M-metric space (X, m) be M-complete and $F : X \to \mathcal{CB}^m(X)$ be a multivalued mapping. Suppose there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$\mathcal{H}_m(Fa, Fb) \leq \lambda[m(a, Fa) + m(b, Fb)], \tag{14}$$

for all $a, b \in X$. Then F admits a fixed point in X.

Proof. Let $a_0 \in X$ be arbitrary. Fix an element $a_1 \in Fa_0$. We can now choose $a_2 \in Fa_1$ such that

$$m(a_1, a_2) = m(a_1, Fa_1) \leq \mathcal{H}_m(Fa_0, Fa_1).$$

Again, we can choose $a_3 \in Fa_2$ such that

$$m(a_2, a_3) \leq \mathcal{H}_m(Fa_1, Fa_2).$$

Continuing in this way, we get a sequence $\{a_k\}$ such that $a_{k+1} \in Fa_k$ with

$$m(a_k, a_{k+1}) \leq \mathcal{H}_m(Fa_{k-1}, Fa_k). \tag{15}$$

Using (14) in (15), we get

$$\begin{aligned} m(a_k, a_{k+1}) &\leq \lambda[m(a_{k-1}, Fa_{k-1}) + m(a_k, Fa_k)] \\ &\leq \lambda[m(a_{k-1}, a_k) + m(a_k, a_{k+1})]. \end{aligned}$$

Thus,

$$m(a_k, a_{k+1}) \leq \frac{\lambda}{1-\lambda} m(a_{k-1}, a_k).$$

Let $r = \frac{\lambda}{1-\lambda}$. Since $\lambda < \frac{1}{2}$, we have $r < 1$. So,

$$m(a_k, a_{k+1}) \leq r m(a_{k-1}, a_k). \tag{16}$$

Thus, from Lemma 5, we have

$$\lim_{k \to \infty} m(a_k, a_{k+1}) = 0, \tag{17}$$

$$\lim_{k \to \infty} m(a_k, a_k) = 0, \tag{18}$$

and

$$\lim_{k,j \to \infty} m(a_k, a_j) = 0. \tag{19}$$

Moreover, the sequence $\{a_k\}$ is a M-Cauchy. M-completeness of X yields existence of $a^* \in X$ such that

$$\lim_{k \to \infty} (m(a_k, a^*) - m_{a_k a^*}) = 0 \text{ and } \lim_{k \to \infty} (M_{a_k a^*} - m_{a_k a^*}) = 0.$$

Due to (18), we get

$$\lim_{k \to \infty} m(a_k, a^*) = 0 \text{ and } \lim_{k \to \infty} M_{a_k a^*} = 0.$$

Thus, we have

$$\lim_{k \to \infty} [M_{a_k a^*} + m_{a_k a^*}] = 0.$$

This implies that

$$m(a^*, a^*) = 0 \text{ and hence } m_{a^* Fa^*} = 0. \tag{20}$$

We shall show that $a^* \in Fa^*$. Since

$$m(a_{k+1}, Fa^*) \leq \mathcal{H}_m(Fa_k, Fa^*) \leq \lambda[m(a_k, Fa_k) + m(a^*, Fa^*)].$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} m(a_{k+1}, Fa^*) = 2\lambda m(a^*, Fa^*). \tag{21}$$

Suppose $m(a^*, Fa^*) > 0$, then we have

$$m(a^*, Fa^*) - m_{a^*Fa^*} \leq m(a^*, a_{k+1}) - m_{a^*a_{k+1}} + m(a_{k+1}, Fa^*) - m_{a_{k+1}Fa^*}.$$

Taking limit as $k \to \infty$ and using (21), we get $m(a^*, Fa^*) \leq 2\lambda m(a^*, Fa^*)$, which is a contradiction (as $2\lambda < 1$). So

$$m(a^*, Fa^*) = 0. \tag{22}$$

Also, using (20), we have

$$\sup_{b\in Fa} m_{a^*b} = \sup_{b\in Fa} \min\{m(a^*, a^*), m(b, b)\} = 0. \tag{23}$$

From (22) and (23), we get

$$m(a^*, Fa^*) = \sup_{b\in Fa} m_{a^*b}.$$

Thus, from Lemma 6, we get $a^* \in \overline{Fa^*} = Fa^*$. $\square$

Example 6. *Let $X = [0, 1]$ and $m : X \times X \to [0, \infty)$ be defined as*

$$m(a, b) = \frac{a + b}{2}.$$

Then (X, m) is an M-complete M-metric space. Let $F : X \to \mathcal{CB}^m(X)$ be a mapping defined as

$$F(a) = \begin{cases} [0, a^2] & \text{if } a \in [0, \frac{1}{2}], \\ \left[\frac{a}{3}, \frac{a}{2}\right] & \text{if } a \in [\frac{1}{2}, 1]. \end{cases}$$

Then one can easily verify that there exists some λ in $(0, \frac{1}{2})$ such that

$$\mathcal{H}_m(Fa, Fb) \leq \lambda\big[m(a, Fa) + m(b, Fb)\big].$$

Thus F satisfies all the conditions in Theorem 2 and hence it has a fixed point (namely 0) in X.

Example 7. *Let $X = [0, 1]$ be endowed with m-metric $m(x, y) = \frac{x+y}{2}$. Then (X, m) is an M-complete M-metric space. We define the mapping $F : X \to \mathcal{CB}^m(X)$ as*

$$F(a) = \begin{cases} \{\frac{1}{5}\} & \text{if } a = 0, \\ \left[\frac{a}{8(1 + a^2)}, \frac{a}{4(1 + a^2)}\right] & \text{if } a > 0. \end{cases}$$

For $a = 0$ and $b = \frac{1}{10}$, there does not exist any λ in $(0, \frac{1}{2})$ such that

$$\mathcal{H}_m(F(0), F(\frac{1}{10})) \leq \lambda\big[m(0, F(0)) + m(\frac{1}{10}, F(\frac{1}{10}))\big].$$

Thus F does not satisfy (14) in Theorem 2. Evidently, F has no fixed point in X.

5. Homotopy Results in M-Metric Spaces

The following result is required in the sequel while proving a homotopy result in M-metric spaces.

Proposition 3. *Let $F : X \to \mathcal{CB}^m(X)$ be a multivalued mapping satisfying (1) for all a, b in M-metric space (X, m). If $c \in Fc$ for some $c \in X$, then $m(a, a) = 0$ for $a \in Fc$.*

Proof. Let $c \in Fc$. Then $m(c, Fc) = \sup\limits_{b \in Fc} m_{c,b} = \sup\limits_{b \in Fc} m_{bb}$. Also

$$\mathcal{H}_m(Fc, Fc) = \delta_m(Fc, Fc) = \sup_{b \in Fc} m_{bb}.$$

Assume that $m(c, c) > 0$. We have

$$\sup_{b \in Fc} m_{bb} = \mathcal{H}_m(Fc, Fc) \leq \lambda m(c, c),$$

that is,

$$\sup_{b \in Fc} m_{bb} \leq \lambda m(c, c).$$

Since $c \in Fc$, it is a contradiction. So $m(a, a) = 0$ for every $a \in Fc$. □

Theorem 3. *Let $\mathcal{O}$ (resp. $\mathcal{C}$) be an open (resp. closed) subset in an M-complete M-metric space (X, m) such that $\mathcal{O} \subset \mathcal{C}$. Let $\mathcal{G} : \mathcal{C} \times [\mu, \nu] \to \mathcal{CB}^m(X)$ be a mapping satisfying the following conditions:*

(a) $a \notin \mathcal{G}(a, t)$ for all $a \in \mathcal{C} \setminus \mathcal{O}$ and each $t \in [\mu, \nu]$;
(b) there exists $\lambda \in (0, 1)$ such that for every $t \in [\mu, \nu]$ and all $a, b \in \mathcal{C}$ we have

$$\mathcal{H}_m(\mathcal{G}(a, t), \mathcal{G}(b, t)) \leq \lambda m(a, b);$$

(c) there exists a continuous mapping $\psi : [\mu, \nu] \to \mathbb{R}$ satisfying

$$\mathcal{H}_m(\mathcal{G}(a, t), \mathcal{G}(a, s)) \leq \lambda |\psi(t) - \psi(s)|;$$

(d) if $c \in \mathcal{G}(c, t)$ then $\mathcal{G}(c, t) = \{c\}$.

If $\mathcal{G}(., t_1)$ admits a fixed point in $\mathcal{C}$ for at least one $t_1 \in [\mu, \nu]$, then $\mathcal{G}(., t)$ admits a fixed point in $\mathcal{O}$ for all $t \in [\mu, \nu]$. Moreover, the fixed point of $\mathcal{G}(., t)$ is unique for any fixed $t \in [\mu, \nu]$.

Proof. Consider, the set

$$\mathcal{W} = \{t \in [\mu, \nu] | a \in \mathcal{G}(a, t) \text{ for some } a \in \mathcal{O}\}.$$

Then $\mathcal{W}$ is nonempty, because $\mathcal{G}(., t_1)$ has a fixed point in $\mathcal{C}$ for at least one $t_1 \in [\mu, \nu]$, that is, there exists $a \in \mathcal{C}$ such that $a \in \mathcal{G}(a, t_1)$ and as (a) holds, we have $a \in \mathcal{O}$.

We will show that $\mathcal{W}$ is both closed and open in $[\mu, \nu]$. First, we show that it is open.

Let $t_0 \in \mathcal{W}$ and $a_0 \in \mathcal{O}$ with $a_0 \in \mathcal{G}(a_0, t_0)$. As $\mathcal{O}$ is open subset of X, $B_m(a_0, r) \subseteq \mathcal{O}$ for some $r > 0$. Let $\varepsilon = r + m_{aa_0} - \lambda(r + m_{aa_0}) > 0$. As ψ is continuous on $[\mu, \nu]$, there exists $\delta > 0$ such that

$$|\psi(t) - \psi(t_0)| < \epsilon, \text{ for all } t \in S_\delta(t_0),$$

where $S_\delta(t_0) = (t_0 - \delta, t_0 + \delta)$.

Since $a_0 \in \mathcal{G}(a_0, t_0)$, by Proposition 3, $m(c,c) = 0$ for every $c \in \mathcal{G}(a_0, t_0)$. Keeping this fact in view, we have

$$m_{pc} = 0, \text{ for every } p \in X. \tag{24}$$

Now, using (iii) of Proposition 2 and (24), we have

$$\begin{aligned}
m(\mathcal{G}(a,t), a_0) &= \mathcal{H}_m(\mathcal{G}(a,t), \mathcal{G}(a_0,t_0)) \\
&\leq \mathcal{H}_m(\mathcal{G}(a,t), \mathcal{G}(a,t_0)) + \mathcal{H}_m(\mathcal{G}(a,t_0), \mathcal{G}(a_0,t_0)) \\
&\quad - \inf_{p \in \mathcal{G}(a,t)} \inf_{q \in \mathcal{G}(a,t_0)} m_{pq} - \inf_{q \in \mathcal{G}(a,t_0)} \inf_{c \in \mathcal{G}(a_0,t_0)} m_{qc} + \sup_{p \in \mathcal{G}(a,t)} \sup_{c \in \mathcal{G}(a_0,t_0)} m_{pc} \\
&\leq \mathcal{H}_m(\mathcal{G}(a,t), \mathcal{G}(a,t_0)) + \mathcal{H}_m(\mathcal{G}(a,t_0), \mathcal{G}(a_0,t_0)) \\
&\leq \lambda|\psi(t) - \psi(t_0)| + \lambda m(a, a_0) \\
&\leq \lambda\varepsilon + \lambda(m_{aa_0} + r) \\
&= \lambda(r + m_{aa_0} - \lambda(r + m_{aa_0})) + \lambda(m_{aa_0} + r) \\
&\leq r + m_{aa_0} - \lambda(r + m_{aa_0}) + \lambda(m_{aa_0} + r) \\
&\leq r + m_{aa_0}.
\end{aligned}$$

Thus for each fixed $t \in S_\delta(t_0)$, $\mathcal{G}(.,t) : \overline{B_m(a_0,r)} \to \mathcal{CB}^m(\overline{B_m(a_0,r)})$ satisfies all the hypotheses of Theorem 1 and so $\mathcal{G}(.,t)$ admits a fixed point in $\overline{B_m(a_0,r)} \subseteq \mathcal{C}$. But this fixed point must be in $\mathcal{O}$ to satisfy (a). Therefore, $S_\delta(t_0) \subseteq \mathcal{W}$ and hence $\mathcal{W}$ is open in $[\mu, \nu]$.

Next, we show that $\mathcal{W}$ is closed in $[\mu, \nu]$. Let $\{t_k\}$ be a convergent sequence in $\mathcal{W}$ to some $s \in [\mu, \nu]$. We need to show that $s \in \mathcal{W}$.

The definition of the set $\mathcal{W}$ implies that for all $k \in \mathbb{N} \setminus \{0\}$, there exists $a_k \in \mathcal{O}$ with $a_k \in \mathcal{G}(a_k, t_k)$. Then using (d), (iii) of Proposition 2 and the outcome of Proposition 3, we have

$$\begin{aligned}
m(a_k, a_j) &= \mathcal{H}_m(\mathcal{G}(a_k,t_k), \mathcal{G}(a_j,t_j)) \\
&\leq \mathcal{H}_m(\mathcal{G}(a_k,t_k), \mathcal{G}(a_k,t_j)) + \mathcal{H}_m(\mathcal{G}(a_k,t_j), \mathcal{G}(a_j,t_j)) \\
&\leq \lambda|\psi(t_k) - \psi(t_j)| + \lambda m(a_k, a_j).
\end{aligned}$$

This gives us

$$m(a_k, a_j) \leq \frac{\lambda}{1-\lambda} \quad |\psi(t_k) - \psi(t_j)|.$$

Since ψ is continuous and $\{t_k\}$ converges to s, varying $k, j \to \infty$ in the above inequality, we get

$$\lim_{k,j \to \infty} m(a_k, a_j) = 0.$$

As $m_{a_k a_j} \leq m(a_k, a_j)$, so

$$\lim_{k,j \to \infty} m_{a_k a_j} = 0.$$

Also $\lim_{k \to \infty} m(a_k, a_k) = 0 = \lim_{k \to \infty} m(a_j, a_j)$.

Therefore

$$\lim_{k,j \to \infty} (m(a_k, a_j) - m_{a_k a_j}) = 0 \text{ and } \lim_{k,j \to \infty} (M_{a_k a_j} - m_{a_k a_j}) = 0.$$

Thus $\{a_k\}$ is an M-Cauchy sequence. Using (iii) of Definition 3, there exists $a^* \in X$ such that

$$\lim_{k \to \infty} (m(a_k, a^*) - m_{a_k a^*}) = 0 \text{ and } \lim_{k \to \infty} (M_{a_k, a^*} - m_{a_k a^*}) = 0.$$

But $\lim_{k\to\infty} m(a_k, a_k) = 0$, so

$$\lim_{k\to\infty} m(a_k, a^*) = 0 \text{ and } \lim_{k\to\infty} M_{a_k a^*} = 0.$$

Thus, we get $m(a^*, a^*) = 0$. We shall prove $a^* \in \mathcal{G}(a^*, t^*)$. We have

$$\begin{aligned} m(a_k, \mathcal{G}(a^*, t^*)) &\leq \mathcal{H}_m(\mathcal{G}(a_k, t_k), \mathcal{G}(a^*, t^*)) \\ &\leq \mathcal{H}_m(\mathcal{G}(a_k, t_k), \mathcal{G}(a_k, t^*)) + \mathcal{H}_m(\mathcal{G}(a_k, t^*), \mathcal{G}(a^*, t^*)) \\ &\leq \lambda|\psi(a_k) - \psi(t^*)| + \lambda m(a_k, a^*). \end{aligned}$$

Varying $k \to \infty$ in above inequality, we get

$$\lim_{k\to\infty} m(a_k, \mathcal{G}(a^*, t^*)) = 0.$$

Hence

$$m(a^*, \mathcal{G}(a^*, t^*)) = 0. \tag{25}$$

Since $m(a^*, a^*) = 0$, we have

$$\sup_{b\in\mathcal{G}(a^*,t^*)} m_{a^*b} = \sup_{b\in\mathcal{G}(a^*,t^*)} \min\{m(a^*, a^*), m(b, b)\} = 0. \tag{26}$$

From (25) and (26), we get

$$m(a^*, \mathcal{G}(a^*, t^*)) = \sup_{b\in\mathcal{G}(a^*,t^*)} m_{a^*b}.$$

Therefore, from Lemma 6, we have $a^* \in \mathcal{G}(a^*, t^*)$. Thus $a^* \in \mathcal{O}$. Hence $t^* \in \mathcal{W}$ and $\mathcal{W}$ is closed in $[\mu, \nu]$.

As $[\mu, \nu]$ is connected and $\mathcal{W}$ is both open and closed in it, so $\mathcal{W} = [\mu, \nu]$. Thus $\mathcal{G}(., t)$ admits a fixed point in $\mathcal{O}$ for all $t \in [\mu, \nu]$.

For uniqueness, fix $t \in [\mu, \nu]$, then there exists $a \in \mathcal{O}$ such that $a \in \mathcal{G}(a, t)$. Suppose b is another fixed point of $\mathcal{G}(b, t)$, then from (d) we have

$$m(a, b) = \mathcal{H}_m(\mathcal{G}(a, t), \mathcal{G}(b, t)) \leq \lambda m(a, b),$$

a contradiction. Thus, the fixed point of $\mathcal{G}(., t)$ is unique for any $t \in [\mu, \nu]$. □

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