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Some New Fractional-Calculus Connections between Mittag–Leffler Functions

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Abstract: We consider the well-known Mittag–Leffler functions of one, two and three parameters, and establish some new connections between them using fractional calculus. In particular, we express the three-parameter Mittag–Leffler function as a fractional derivative of the two-parameter Mittag–Leffler function, which is in turn a fractional integral of the one-parameter Mittag–Leffler function. Hence, we derive an integral expression for the three-parameter one in terms of the one-parameter one. We discuss the importance and applications of all three Mittag–Leffler functions, with a view to potential applications of our results in making certain types of experimental data much easier to analyse.

Keywords: fractional integrals; fractional derivatives; Mittag–Leffler functions

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1. Introduction

In fractional calculus, the standard calculus operations of differentiation and integration are generalised to orders beyond the integers: rational, real, and even complex numbers can be used for the order of *differintegration* [1–3]. This area of research is four centuries old, but it has expanded rapidly only in the last fifty years, discovering applications in many fields of science and engineering [4–6]. The most commonly used definition of fractional derivatives and integrals is the Riemann–Liouville one, where fractional integrals are defined by

$$ {}^{RL}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-y)^{-\alpha-1} f(y)\,\mathrm{d}y, \qquad \mathrm{Re}(\alpha) < 0, \tag{1} $$

and fractional derivatives are defined by

$$ {}^{RL}_{a}D^{\alpha}_{x}f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left({}^{RL}_{a}D^{\alpha-m}_{x} f(x)\right), \qquad \mathrm{Re}(\alpha) \geq 0, m := \lfloor \mathrm{Re}(\alpha) \rfloor + 1. \tag{2} $$

Here, $D^{\alpha} f$ denotes the derivative to order α of a function f, and a is a constant of differintegration. It is important to note that in fractional calculus, derivatives as well as integrals rely on the choice of an arbitrary constant a. This constant is usually set to be either $a = 0$ or $a = -\infty$. To see why both choices are useful, we present the following Lemma which provides two "'natural" differintegration formulae, one requiring $a = 0$ and the other requiring $a = -\infty$. Neither option can be eliminated from the range of possible values for a, if we wish to retain natural expressions for differintegrals of elementary functions.

Lemma 1. *The Riemann–Liouville (RL) differintegrals of power functions and exponential functions, with constant of differintegration $a = 0$ and $a = -\infty$, respectively, are as follows.*

$$ {}^{RL}_{0}D^{\alpha}_{x}(x^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{\beta-\alpha}, \qquad \alpha,\beta \in \mathbb{C}, \mathrm{Re}(\beta) > -1; \tag{3} $$

$$ {}^{RL}_{-\infty}D^{\alpha}_{x}\left(e^{\beta x}\right) = \beta^{\alpha} e^{\beta x}, \qquad \alpha,\beta \in \mathbb{C}, \beta \notin \mathbb{R}^{-}_{0}. \tag{4} $$

In both cases, we define complex power functions using the principal branch with arguments between $-\pi$ and π.

Proof. In both cases, the proof for fractional integrals follows from manipulation and substitution in the integral formula in Equation (1), and then the proof for fractional derivatives is immediate from the definition in Equation (2). For more details, we refer the reader to [1,3]. □

In recent years, many alternative definitions of fractional differintegrals have been proposed. Some of these were motivated by the different real-world systems which can be modelled by different fractional-calculus structures: for example, replacing the power function in Equation (1) by another function to better describe certain types of processes in dynamical systems [7,8]. Others were created by adding extra parameters and levels of generalisation into functions and formulae [9–11].

One particular function which frequently appears in the study of fractional derivatives and integrals [12–14] is the **Mittag–Leffler function**, which in its simplest form is defined by

$$ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \qquad z \in \mathbb{C}. \tag{5} $$

The above function with a single parameter α has also been extended to more general functions defined with two or more parameters, such as the following [15,16].

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \qquad z \in \mathbb{C}; \tag{6} $$

$$ E^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)z^n}{\Gamma(\rho)\Gamma(n\alpha+\beta)n!}, \qquad z \in \mathbb{C}; \tag{7} $$

$$ E^{\rho,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)z^n}{\Gamma(\rho)\Gamma(n\alpha+\beta)n!}, \qquad z \in \mathbb{C}. \tag{8} $$

It is clear that the following interrelations hold between the above functions:

$$ E^{\rho,1}_{\alpha,\beta}(z) = E^{\rho}_{\alpha,\beta}(z), \qquad E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), \qquad E_{\alpha,1}(z) = E_{\alpha}(z). $$

Several new models of fractional calculus have used such functions in their definitions, and we mention two of these in particular.

The **AB model**, formulated by Atangana and Baleanu [7] and further studied in [17–20], is defined by replacing the power function in Equation (1) by a one-parameter Mittag–Leffler function of the type in Equation (5):

$$ {}^{ABR}_{a}D^{\alpha}_{x}f(x) = \frac{B(\alpha)}{1-\alpha} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(x-y)^{\alpha} \right) f(y)\,\mathrm{d}y, \qquad 0 < \alpha < 1, \tag{9} $$

$$ {}^{AB}_{a}I^{\alpha}_{x}f(x) = \frac{1-\alpha}{B(\alpha)}f(x) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{x}f(x), \qquad 0 < \alpha < 1. \tag{10} $$

We note that here α is a real variable and not a complex one. All discussion of the AB model in the literature so far has assumed the order of differentiation to be real. The first paper to consider complex-order AB differintegrals is currently in press [21].

The **Prabhakar model**, based on an integral operator defined in 1971 [22] but only later formulated as part of fractional calculus [23,24], is defined by replacing the power function in Equation (1) by a three-parameter Mittag–Leffler function of the type in Equation (7). This model has also been generalised [16] to use a 4-parameter Mittag–Leffler function of the type in Equation (8), and its properties have been explored in many papers (e.g., [25–27]).

One useful application of fractional calculus in pure mathematics has been to find new functional equations and interrelations between various important functions. For example, fractional versions of the product rule and chain rule have given rise to new formulae for assorted special functions [28,29], and fractional differintegration of infinite series has yielded new identities on the Riemann zeta function and its generalisations [30–32].

In the current work, we use these techniques to prove new relationships between the several Mittag–Leffler functions defined above. It is possible to write the three-parameter Mittag–Leffler function as a fractional differintegral of the two-parameter one, as well as writing the two-parameter one as a fractional differintegral of the one-parameter one, and thence to deduce an integral relationship between the one-parameter and three-parameter Mittag–Leffler functions, which suggests a relationship between the AB and Prabhakar models of fractional calculus. Throughout all of this, we use only the classical Riemann–Liouville fractional integrals and derivatives. We also examine the possibility of applications of these results in fields of science such as bioengineering and dielectric relaxation.

This paper is structured as follows. In Section 2, through a number of theorems and propositions, we state the main results concerning relationships between Mittag–Leffler functions. In Section 3, we discuss applications, and, in Section 4, we conclude the article.

2. The Main Results

We first state an important result about fractional differintegration of series, which we need to use in the proofs below.

Lemma 2. *Consider a function S defined by an infinite series*

$$ S(x) = \sum_{n=1}^{\infty} S_n(x) $$

which is uniformly convergent on the set $|x-a| \leq K$ for some fixed constants $a \in \mathbb{C}$, $K > 0$. Let $\alpha \in \mathbb{C}$ be a fixed order of differintegration.

1. *If* $\mathrm{Re}(\alpha) < 0$ *(fractional integration), then we have*

$$ {}^{RL}_{a}D^{\alpha}_{x}S(x) = \sum_{n=1}^{\infty} {}^{RL}_{a}D^{\alpha}_{x}S_n(x), \qquad |x-a| \leq K, $$

and the series on the right-hand side is uniformly convergent on the given region.

2. *If* $\mathrm{Re}(\alpha) \geq 0$ *(fractional differentiation) and the series* $\sum_{n=1}^{\infty} {}^{RL}_{a}D^{\alpha}_{x}S_n(x)$ *is uniformly convergent on the region* $|x-a| \leq K$*, then we have*

$$ {}^{RL}_{a}D^{\alpha}_{x}S(x) = \sum_{n=1}^{\infty} {}^{RL}_{a}D^{\alpha}_{x}S_n(x), \qquad |x-a| \leq K. $$

Proof. This is Theorem VIII in [31]. □

Since our work involves analytic functions defined on the complex plane, we find it useful to define the domain

$$ \mathbb{D} := \mathbb{C} \backslash \mathbb{R}_0^-, \tag{11} $$

namely the complex plane slit along a branch cut from the origin. This is used as a domain for various complex power functions and other related functions.

Our first main result is a fractional identity between the Mittag–Leffler functions with two and three parameters as defined by Equations (6) and (7). This is motivated by previous work (e.g., [32]), in which gamma functions that appear in infinite power series can be interpreted as arising from fractional differintegrals. After submitting the paper, we realised that this result was previously proved in [33]. However, our original proof is preserved below.

Proposition 1. *For any* $\alpha, \beta, \rho \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$*, we have:*

$$ E^{\rho}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\rho)} {}^{RL}_{0}D^{\rho-1}_{z} \left[z^{\rho-1} E_{\alpha,\beta}(z) \right], \qquad z \in \mathbb{D}. \tag{12} $$

Proof. First, Lemma 1 tells us that a quotient of gamma functions can very often be interpreted as arising from a fractional differintegral of a power function. In this case, the expression $\frac{\Gamma(\rho+n)}{n!}$ appearing in the coefficients of the series in Equation (7) gives rise to the following:

$$ \begin{aligned} E^{\rho}_{\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n) z^n}{\Gamma(\rho)\Gamma(n\alpha+\beta) n!} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+\beta)\Gamma(\rho)} \cdot \frac{\Gamma(\rho+n)}{\Gamma(n+1)} z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+\beta)\Gamma(\rho)} \cdot {}^{RL}_{0}D^{\rho-1}_{z} \left[z^{n+\rho-1} \right] = \frac{1}{\Gamma(\rho)} \sum_{n=0}^{\infty} {}^{RL}_{0}D^{\rho-1}_{z} \left[\frac{z^{n+\rho-1}}{\Gamma(n\alpha+\beta)} \right]. \end{aligned} $$

Since the series here is uniformly convergent, we can use the result of Lemma 2 to swap the summation with the fractional differintegration. This gives:

$$ \begin{aligned} E^{\rho}_{\alpha,\beta}(z) &= \frac{1}{\Gamma(\rho)} \sum_{n=0}^{\infty} {}^{RL}_{0}D^{\rho-1}_{z} \left[\frac{z^{n+\rho-1}}{\Gamma(n\alpha+\beta)} \right] = \frac{1}{\Gamma(\rho)} {}^{RL}_{0}D^{\rho-1}_{z} \left[\sum_{n=0}^{\infty} \frac{z^{n+\rho-1}}{\Gamma(n\alpha+\beta)} \right] \\ &= \frac{1}{\Gamma(\rho)} {}^{RL}_{0}D^{\rho-1}_{z} \left[z^{\rho-1} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} \right] = \frac{1}{\Gamma(\rho)} {}^{RL}_{0}D^{\rho-1}_{z} \left[z^{\rho-1} E_{\alpha,\beta}(z) \right], \end{aligned} $$

and we have the result desired. □

Note that, by setting $\rho = 1$ in Equation (12), we recover the trivial identity $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

Corollary 1. *For any* $\alpha, \rho \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$*, we have:*

$$ E^{\rho}_{\alpha,1}(z) = \frac{1}{\Gamma(\rho)} {}^{RL}_{0}D^{\rho-1}_{z} \left[z^{\rho-1} E_{\alpha}(z) \right], \qquad z \in \mathbb{D}. \tag{13} $$

Proof. This follows immediately by setting $\beta = 1$ in Proposition 1. □

Remark 1. *By exactly the same argument as in Proposition 1, we can show that*

$$E^{\rho}_{\alpha,\beta}(\gamma z) = \frac{1}{\Gamma(\rho)}\, {}^{RL}_{\ 0}D^{\rho-1}_{z}\left[z^{\rho-1}E_{\alpha,\beta}(\gamma z)\right], \qquad z \in \mathbb{D}, \tag{14}$$

for any $\alpha, \beta, \gamma, \rho \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. The proof is as above with an extra factor of γ^n included in each term of the sum.

We have now established a relation between the Mittag–Leffler functions of two and three parameters, and hence a relation between the Mittag–Leffler function of one parameter and the function $E^{\rho}_{\alpha,1}(z)$. However, in order to find a connection with the AB model, we need to consider not the function $E_\alpha(x)$ but rather the function $E_\alpha\left(\frac{-\alpha}{1-\alpha}x^\alpha\right)$, which appears in the kernel of the definition in Equation (9). To this end, we note the following result, which is seen in (Equation (7.1), [12]) but without reference to fractional calculus.

Proposition 2. *For any $\alpha, \beta, \gamma \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, we have:*

$$E_{\alpha,\beta}\left(\gamma z^\alpha\right) = z^{1-\beta}\, {}^{RL}_{\ 0}D^{1-\beta}_{z}\left[E_\alpha\left(\gamma z^\alpha\right)\right], \qquad z \in \mathbb{D}. \tag{15}$$

Proof. This time we start from the right-hand side of the desired identity, and use the definition in Equation (5) of the function E_α:

$$z^{1-\beta}\, {}^{RL}_{\ 0}D^{1-\beta}_{z}\left[E_\alpha\left(\gamma z^\alpha\right)\right] = z^{1-\beta}\, {}^{RL}_{\ 0}D^{1-\beta}_{z}\left[\sum_{n=0}^{\infty}\frac{\gamma^n z^{\alpha n}}{\Gamma(n\alpha+1)}\right].$$

This series is uniformly convergent, thus, by Lemma 2, we can swap the summation and fractional differintegration provided that (at least in the case $0 < \mathrm{Re}(\beta) < 1$) the resulting series also converges uniformly. We swap the operations now and justify this assumption at the end.

$$\begin{aligned}
z^{1-\beta}\, {}^{RL}_{\ 0}D^{1-\beta}_{z}\left[E_\alpha\left(\gamma z^\alpha\right)\right] &= z^{1-\beta}\sum_{n=0}^{\infty}{}^{RL}_{\ 0}D^{1-\beta}_{z}\left[\frac{\gamma^n z^{\alpha n}}{\Gamma(n\alpha+1)}\right] \\
&= z^{1-\beta}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+\beta)}\cdot\frac{\gamma^n z^{\alpha n+\beta-1}}{\Gamma(n\alpha+1)} \\
&= \sum_{n=0}^{\infty}\frac{\gamma^n z^{\alpha n}}{\Gamma(\alpha n+\beta)}.
\end{aligned}$$

This series converges uniformly, being precisely the series expression in Equation (6) for $E_{\alpha,\beta}\left(\gamma z^\alpha\right)$. Thus, our swapping of operations above was justified, and the proof is complete. □

The key point here is that the dependence of the Mittag–Leffler function in Equation (7) on the parameters ρ and β can be encoded by fractional differintegrals. Proposition 1 enables us to interpret the parameter ρ as merely the order of a differintegral, and Proposition 2 enables us to do the same with β.

By combining the results of Proposition 1 and Proposition 2, it is possible to obtain a composite expression for the three-parameter Mittag–Leffler function in Equation (7) in terms of fractional-type integrals, as described by the following theorem.

Theorem 1. *For any $\alpha, \beta, \gamma, \rho \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\rho) < 1$, we have:*

$$E^{\rho}_{\alpha,\beta}(\gamma z^\alpha) = \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^z \left(z^\alpha - u^\alpha\right)^{-\rho} u^{\alpha\rho-\beta}\, {}^{RL}_{\ 0}D^{1-\beta}_{u}\left[E_\alpha\left(\gamma u^\alpha\right)\right]\mathrm{d}u, \qquad z \in \mathbb{D}. \tag{16}$$

Proof. For $\mathrm{Re}(\rho) < 1$, the fractional differintegrals appearing in Proposition 1 and Corollary 1 are *integrals* (because $\mathrm{Re}(\rho-1) < 0$), and so Equation (14) can be rewritten as follows:

$$E^{\rho}_{\alpha,\beta}(\gamma z) = \frac{1}{\Gamma(\rho)}\,{}^{RL}_{0}I^{1-\rho}_{z}\left[z^{\rho-1}E_{\alpha,\beta}(\gamma z)\right] = \frac{1}{\Gamma(\rho)\Gamma(1-\rho)}\int_0^z (z-y)^{-\rho}y^{\rho-1}E_{\alpha,\beta}(\gamma y)\,\mathrm{d}y.$$

Making the change of variables $u = y^{1/\alpha}$, and using the reflection formula for the gamma function:

$$\begin{aligned} E^{\rho}_{\alpha,\beta}(\gamma z) &= \frac{\sin(\pi\rho)}{\pi}\int_0^{z^{1/\alpha}} (z-u^{\alpha})^{-\rho}\,u^{\alpha(\rho-1)}E_{\alpha,\beta}(\gamma u^{\alpha})\alpha u^{\alpha-1}\,\mathrm{d}u \\ &= \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{z^{1/\alpha}} (z-u^{\alpha})^{-\rho}\,u^{\alpha\rho-1}E_{\alpha,\beta}(\gamma u^{\alpha})\,\mathrm{d}u. \end{aligned}$$

Now, we can apply the result of Proposition 2 to the two-parameter Mittag–Leffler function appearing in the integrand of this expression:

$$\begin{aligned} E^{\rho}_{\alpha,\beta}(\gamma z) &= \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{z^{1/\alpha}} (z-u^{\alpha})^{-\rho}\,u^{\alpha\rho-1}\left[u^{1-\beta}\,{}^{RL}_{0}D^{1-\beta}_{u}\left[E_{\alpha}\left(\gamma u^{\alpha}\right)\right]\right]\mathrm{d}u \\ &= \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{z^{1/\alpha}} (z-u^{\alpha})^{-\rho}\,u^{\alpha\rho-\beta}\,{}^{RL}_{0}D^{1-\beta}_{u}\left[E_{\alpha}\left(\gamma u^{\alpha}\right)\right]\mathrm{d}u. \end{aligned}$$

Substituting z^{α} for z:

$$E^{\rho}_{\alpha,\beta}(\gamma z^{\alpha}) = \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{z} (z^{\alpha}-u^{\alpha})^{-\rho}\,u^{\alpha\rho-\beta}\,{}^{RL}_{0}D^{1-\beta}_{u}\left[E_{\alpha}\left(\gamma u^{\alpha}\right)\right]\mathrm{d}u.$$

Note that this can almost, but not quite, be expressed as a composition of two fractional differintegrals. □

Corollary 2. *For any $\alpha,\beta,\gamma,\rho\in\mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 1$, and $\mathrm{Re}(\rho) < 1$, we have:*

$$E^{\rho}_{\alpha,\beta}(\gamma z^{\alpha}) = \frac{\alpha\sin(\pi\rho)}{\pi\Gamma(\beta-1)}\int_0^{z} (z^{\alpha}-u^{\alpha})^{-\rho}\,u^{\alpha\rho-\beta}\int_0^u (u-t)^{\beta-2}E_{\alpha}\left(\gamma t^{\alpha}\right)\mathrm{d}t\,\mathrm{d}u, \qquad z\in\mathbb{D}. \tag{17}$$

Proof. In this case, we have $\mathrm{Re}(1-\beta) < 0$ and so the fractional differintegral which appears in the integrand of Equation (16) is an *integral*. Thus, we can write

$${}^{RL}_{0}I^{\beta-1}_{u}\left[E_{\alpha}\left(\gamma u^{\alpha}\right)\right] = \frac{1}{\Gamma(\beta-1)}\int_0^u (u-t)^{\beta-2}E_{\alpha}\left(\gamma t^{\alpha}\right)\mathrm{d}t,$$

and substitute this into Equation (16) to find:

$$E^{\rho}_{\alpha,\beta}(\gamma z^{\alpha}) = \frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{z} (z^{\alpha}-u^{\alpha})^{-\rho}\,u^{\alpha\rho-\beta}\left[\frac{1}{\Gamma(\beta-1)}\int_0^u (u-t)^{\beta-2}E_{\alpha}\left(\gamma t^{\alpha}\right)\mathrm{d}t\right]\mathrm{d}u,$$

which rearranges to the required result. □

Theorem 2. *The three-parameter Mittag–Leffler function in Equation (7) can be written as an integral transform of the one-parameter Mittag–Leffler function in Equation (5) in the following way:*

$$E^{\rho}_{\alpha,\beta}(\gamma z^{\alpha}) = \frac{\alpha\sin(\pi\rho)}{\pi\Gamma(\beta-1)}\int_0^{z} F_{\alpha,\beta,\rho}(t;z)E_{\alpha}\left(\gamma t^{\alpha}\right)\mathrm{d}t \qquad z\in\mathbb{D}, \tag{18}$$

where we assume $\alpha, \beta, \gamma, \rho \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 1$, *and* $\mathrm{Re}(\rho) < 1$, *and where the function* F *is defined as*

$$F_{\alpha,\beta,\rho}(t;z) := \int_t^z (z^\alpha - u^\alpha)^{-\rho} \, u^{\alpha\rho-\beta}(u-t)^{\beta-2} \, \mathrm{d}u. \tag{19}$$

Proof. By Fubini's theorem, it is possible to swap the order of the integrals in Equation (17). We have $0 \leq u \leq z$ and $0 \leq t \leq u$, which after swapping is equivalent to $0 \leq t \leq z$ and $t \leq u \leq z$. We have from Equation (17):

$$\begin{aligned}
E^{\rho}_{\alpha,\beta}(\gamma z^\alpha) &= \frac{\alpha \sin(\pi\rho)}{\pi\Gamma(\beta-1)} \int_0^z \int_0^u (z^\alpha - u^\alpha)^{-\rho} \, u^{\alpha\rho-\beta}(u-t)^{\beta-2} E_\alpha\left(\gamma t^\alpha\right) \mathrm{d}t \, \mathrm{d}u \\
&= \frac{\alpha \sin(\pi\rho)}{\pi\Gamma(\beta-1)} \int_0^z \int_t^z (z^\alpha - u^\alpha)^{-\rho} \, u^{\alpha\rho-\beta}(u-t)^{\beta-2} E_\alpha\left(\gamma t^\alpha\right) \mathrm{d}u \, \mathrm{d}t \\
&= \frac{\alpha \sin(\pi\rho)}{\pi\Gamma(\beta-1)} \int_0^z E_\alpha\left(\gamma t^\alpha\right) \int_t^z (z^\alpha - u^\alpha)^{-\rho} \, u^{\alpha\rho-\beta}(u-t)^{\beta-2} \, \mathrm{d}u \, \mathrm{d}t \\
&= \frac{\alpha \sin(\pi\rho)}{\pi\Gamma(\beta-1)} \int_0^z E_\alpha\left(\gamma t^\alpha\right) F_{\alpha,\beta,\rho}(t;z) \, \mathrm{d}t,
\end{aligned}$$

as required. □

Thus, using Riemann–Liouville fractional calculus, we have forged new connections between the Mittag–Leffler functions of one, two and three parameters. The connection between those of one and three parameters, in particular, may give rise to new formulae linking AB fractional calculus with Prabhakar fractional calculus, in a way more profound than simply writing one as a special case of the other.

3. Applications

As we have already discussed, the various Mittag–Leffler functions are interesting from the point of view of pure mathematical analysis and fractional calculus [34–37] (see also the correction [16,36,38]). However, it is also important to discuss the motivation for studying these functions from the point of view of real-world applications in science and engineering.

The one-parameter Mittag–Leffler function has already discovered many applications via the AB model, and also previously in relaxation models which involve interpolation between exponential and power-law behaviours [39]. In recent years, the two-parameter and three-parameter Mittag–Leffler functions have also been emerging from real experimental data.

A group of biologists and engineers in Cambridge and London have been experimenting with models for cells and tissues, and discovered that their data fit most closely to an operator involving two-parameter Mittag–Leffler functions [40].

The three-parameter Mittag–Leffler function, sometimes called the Prabhakar function, is closely connected with the phenomenon of Havriliak–Negami relaxation [41], and this has been studied also in the context of fractional relaxation [26,42].

In view of these manifold applications of the Mittag–Leffler functions of one, two and three parameters, we believe that our results herein may also discover applications. The one-parameter Mittag–Leffler function is much more elementary and easier to handle than the two- and three-parameter Mittag–Leffler functions. Thus, reducing the latter to the former should mark a major step forward. The physical processes which are modelled using two- and three-parameter Mittag–Leffler functions may now be more easily analysed using only the one-parameter Mittag–Leffler function.

In particular, we note that numerical computation of Mittag–Leffler functions has been a challenging problem for researchers in recent years [43–45]. Naturally, the one-parameter Mittag–Leffler function is the most straightforward to handle. If we can use relations such as those proved in this paper to express the more advanced Mittag–Leffler functions purely in terms of the most basic one, then it may enable much easier computation of the two- and three-parameter Mittag–Leffler functions than before.

4. Conclusions

In this article, we have established new relations between the Mittag–Leffler functions of one, two and three parameters by using Riemann–Liouville fractional calculus. The main results are Proposition 1 (three-parameter Mittag–Leffler in terms of two-parameter Mittag–Leffler), and Theorem 2 (three-parameter Mittag–Leffler in terms of one-parameter Mittag–Leffler), which come from combining Proposition 1 with Proposition 2 (two-parameter Mittag–Leffler in terms of one-parameter Mittag–Leffler). We believe that these results can be applied in the future, to simplify some important physical models that use two- or three-parameter Mittag–Leffler functions, or to provide more efficient computational models for these functions, since the original one-parameter Mittag–Leffler function is much better known and more deeply studied.

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