ON THE PARTIAL GENERALIZATION OF THE MEASURE OF TRANSCENDENCE OF SOME FORMAL LAURENT SERIES

Ahmet Ş. ÖZDEMİR
Atatürk Education Faculty, Department of Mathematics
Marmara University, İstanbul / TURKEY

Abstract: In this work, we determine the transcendence measure of the formal Laurent series "that"

$$\xi = \psi(r) = \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{F_k}$$

whose transcendence has been established by L.I.Wade. Using the methods and lemmas in P.Bundschuh's article, measure of the transcendence for the above $\xi$ is determined as

$$T(n, H) = H^{-\alpha d^d(\frac{1}{k} - d - 1)}.$$

On the other hand, it was proven that the transcendence series $\xi$ is not a U but is a S or T-number according to the Mahler's classification.
**INTRODUCTION**

Let $p$ a prime number and $u \geq 1$ an integer. Let $F$ be a finite field with $q = p^u$ elements. We denote the ring of the polynomials with one variable over $F$ by $F[x]$ and its quotient field by $F(x)$. If $a \in F[x]$ is a non-zero polynomial, denote its degree by $\partial a$. If $a = 0$, then its degree is defined as $\partial 0 := -\infty$. Let $a$ and $b$ ($b \neq 0$) two polynomials from $F[x]$ and define a discrete valuation of $F(x)$ as follows

$$|a_b| = q^{\partial a - \partial b}.$$

Let $K$ be the completion of $F(x)$ with respect to this valuation. Every element $\omega$ of $K$ can be uniquely represented by

$$\omega = \sum_{n=k}^{\infty} c_n x^{-n}, c_n \in F.$$

If $\omega = 0$, then all $c_n$ are zero. If $\omega \neq 0$, then there exist an $k \in \mathbb{Z}$ for which $c_k \neq 0$. If $\omega \neq 0$, then we have

$$|\omega| = q^{-k}.$$

Therefore $K$ is the field of all formal Laurent series. The classical theory of transcendence over complex numbers has a similar version over $K$. Elements of $F[x]$ and $F(x)$ correspond to integers and fractions of the classical theory, respectively.

If $\omega \in K$ is one of the roots of a non-zero polynomial with coefficients in $F[x]$, then $\omega \in K$ is said to be algebraic over $F(x)$. Otherwise, $\omega$ is called transcendental over $F(x)$. The studies of transcendental numbers in $K$ were initiated first by Wade [1-4]. Also Geijsel [5-8] did similar studies. As it is the case in the classical theory of transcendental numbers, it is possible to define a measure of transcendence

The measure of transcendence is thoroughly studied in the classical theory. For example, the transcendence measure of $e$ has been widely investigated by Mahler [9] and Fel’dman [10]. Examples for the transcendence measure in the field $K$ have been given for the first time by Bundschuh [12].
In this work, we determine a transcendence measure of some formal Laurent series whose transcendence has been established by L.I.Wade [2]. If \( r^q \) (where \( r \in \text{RealNumbers} \)) and \( F_k \in F[x] \) is a fixed non-zero polynomial of degree \( \partial(r^q) = 0 \) and \( \partial(F_k) = kq^k \) then the series
\[
\xi = \psi(r) = \sum_{k=0}^{\infty} \frac{(-1)^k r^q}{F_k}
\]
is an element of \( K \), and L.I.Wade showed its transcendence in [2]. (see Theorem 3.1 and 3.2) Using the methods and lemmas in Bundschuh’s article [12], we determine a transcendence measure of \( \xi \). We take an arbitrary non-zero polynomial
\[
P(y) = \sum_{\nu=0}^{n} a_\nu y^\nu, (a_\nu \in F[x]; \nu = 0, 1, ..., n)
\]
whose degree \( \partial(P) \) is less than or equal to \( n \). The height of \( P \) is denoted by
\[
h(P) = m_{\nu=0}^{n} |a_\nu| = q_{\nu=0}^{n} |a_\nu|^\partial(\nu)
\]
For the transcendental element \( \xi \) of \( K \), we define the positive quantity
\[
\Gamma_n(H, \xi) = \min |P(\xi)|
\]
where \( P \neq 0, \partial(P) \leq n, h(P) \leq H \).

If \( T(n, H) \) is a function of the variables \( n, H \) of \( \Gamma_n(H, \xi) \) which satisfies the inequality
\[
\Gamma_n(H, \xi) \geq T(n, H)
\]
for all sufficiently large values of \( n \) and \( H \), then \( T(n, H) \) is said to be a transcendence measure of \( \xi \).
Theorem 1:

We take an arbitrary non-zero polynomial

\[ P(y) = \sum_{\nu=0}^{d} a_{\nu} y^{\nu}, (a_{\nu} \in F[x]; \nu = 0, 1, \ldots, n) \quad (4) \]

further let \( \partial(P) = d \), \( h(P) = h \) and \( a = \max_{i=0}^{d} \partial a_{\nu} \). We assume that

\[ dq^{d} \log h > \frac{kq^{k} \log q}{q} \quad (5) \]

we have

\[ |P(\xi)| \geq h^{-aq^{d}(\frac{1}{k} - d - 1)} \quad (6) \]

and a transcendence measure of \( \xi \) is

\[ T(n, H) = H^{-aq^{d}(\frac{1}{k} - d - 1)}. \quad (7) \]

As in the classical theory of transcendental number theory (see Schneider [13], page 6), it is possible to define Mahler’s classification on \( K \). Let \( \xi \in K \) be transcendental, and define:

\[ \Gamma_{n}(\xi) := \lim_{H \to \infty} \sup_{n} \frac{-\log \Gamma_{n}(H, \xi)}{\log H} \]

\[ \Gamma(\xi) = \lim_{H \to \infty} \frac{1}{n} \Gamma_{n}(\xi) \quad (8) \]

Hence \( \Gamma_{n}(\xi) \geq n \) for every \( n \in \mathbb{N} \) and so \( \Gamma(\xi) \geq 1 \). For every \( n, H \in \mathbb{N} \),

\[ \Gamma_{n}(H, \xi) < H^{-n} q^{n} \max(1, |\xi|^{n}) \quad (9) \]

is satisfied (see Bundschuh [12], Lemma 3).

On the other hand, let the least naturel number \( n \) satisfying \( \Gamma_{n}(\xi) = \infty \) be denoted by \( \mu(\xi) \). If there is no such \( n \), then one may define \( \mu(\xi) \) as \( \infty \). In this case, the transcendental number \( \xi \in \mathbb{R} \) is called

- \( S \)-Laurent series if \( 1 \leq \Gamma(\infty) < \infty \) and \( \mu(\xi) = \infty \),
- \( T \)-Laurent series if \( \Gamma(\xi) = \infty \) and \( \mu(\infty) = \infty \),
- \( U \)-Laurent series if \( \Gamma(\xi) = \infty \) and \( \mu(\infty) < \infty \).
Moreover, the U-class may be divided into subclasses. If \( \mu(\xi) = m(m > 0) \) then \( \xi \) is called a \( U_m \)-Laurent series. Leveque [11] was the first to show that for all \( m \), \( U_m \) is non-empty in the classical theory but the honour goes to Oryan [14] if the ground field is \( K \).

According to the above classification, the series defined in (1) can not be a \( U \)-Laurent series. This fact may be proved by the help of the Theorem 1.

**Theorem 2**: The \( \xi \) Laurent series defined by (1) doesn’t belong to the class \( U \) so that it belongs to the class \( S \) or to the class \( T \).

**Preliminary**

We will use the following lemmas in proof of the theorem.

**Lemma 1**: Let

\[
P(y) = \sum_{\nu=0}^{d} a_{\nu} y^{\nu}
\]

\((a_{\nu} \in F[x], a_d \neq 0 (d \geq 1) \), \( a = \frac{d}{\partial x}, \partial a_{\nu} \).

Then there are some elements \( A_0, A_1, ..., A_d \in F[x] \), not all zero satisfying, \( \partial A_j \leq ad(q^d - d + 1) \) for \( 0 \leq j \leq d \) and

\[
\sum_{j=0}^{d} A_j y^{q_j} = p(y) \sum_{j=0, q^j \geq d}^{d} A_j \sum_{k=0}^{q^j-d} b_k a_d^{-k-1} y^{q^j-d-k} =: P(y)Q(y)
\]

where \( b_0 := 1 \) and \( b_k \), for \( k \geq 1 \) is the sum of product of exactly \( k \) terms from \( a_0, a_1, ..., a_d \), multiplied by \( \pm \).

**Proof**: See [12], lemma 4, page 416

**Lemma 2**: Let \( \xi \in K \) and \( |\xi| = q^{\lambda} \). Under the hypotheses of Lemma 1 we have

\[
|Q(\xi)| \leq q^{d(q^d-d+1)+(q^d-d)\max(a, \lambda)}.
\]

**Proof**: See [12], lemma 5, page 417
PROOF OF THE THEOREMS

Proof of the Theorem 1:

Consider the polynomial defined by (4). With \( \theta(p) = d \), \( a_d \neq 0 \). Let \( d \geq 1 \).
By Lemma 1 there are some elements the \( A_0, A_1, \ldots, A_d \in F[x] \) not all zero
such that
\[
\sum_{j=0}^{d} A_j y^{q^j} = P(y) \sum_{j=0}^{d} A_j \sum_{k=0}^{q^j-d} b_k a_d^{-k-1} y^{q^j-d-k} =: P(y)Q(y) \tag{13}
\]
\[
\partial(A_j) \leq ad(q^d - d + 1) \leq adq^d \quad (0 \leq j \leq d) . \tag{14}
\]

In (13) we put \( \xi \) instead of \( y \):
\[
P(\xi)Q(\xi) = \sum_{j=0}^{d} A_j \xi^{q^j} = \sum_{j=0}^{d} A_j \sum_{k=0}^{\infty} (-1)^{q^j+k} F_{q^j}^{-q^j} . \tag{15}
\]

Furthermore let \( D_k = \sum_{k=i+j}^{\infty} (-1)^{i+j} \frac{F_{q^j}^{-q^j} F_k}{F_j q^j} \)
Separate in (12) sum as \( T_1 + T_2 \), where
\[
T_1 = F_{\beta} \sum_{k=0}^{\beta} \frac{D_k}{F_k}, \quad T_2 = F_{\beta} \sum_{k=\beta+1}^{\infty} \frac{D_k}{F_k} \tag{16}
\]
where \( \beta \), which is not a negative integer will be chosen later.

1) First, we prove that \(|T_1| \geq 1\). That is, we prove \( T_1 \) is a polynomial
but not equal zero. By the definition of \( F_k \), obviously \( T_1 \) is polynomial.
Furthermore,
\[
T_1 \equiv D_\beta(\mod[\beta - l]) \\
\equiv (-1)^{\beta-l} A_i[\beta] \ldots[\beta - l+1]^{l-1} \\
\equiv (-1)^{\beta-l} A_i F_i \neq 0(\mod[\beta - l])
\]

for \( \beta \) sufficiently large. Therefore, for all sufficiently \( \beta \) \( T_1 \) is not identically
zero. So \( T_1 \) is non-zero polynomial. So it shown that \(|T_1| \geq 1\). (where
\( \deg T_1 > 0 \implies |T_1| \geq 1 = q^{\deg T_1} > q^0 = 1 \)

2) we will show \(|T_2| < 1\). Let \( T_2^* \) be any term of \( T_2 \). Note that ,

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\[ \text{deg } D_{\beta} = \text{deg } F_{\beta} + \text{deg } A_j - \text{deg } F_{\beta-j}^{q^j} + \text{deg } r^q \]

where \( r^q \neq 0 \).

\[ \text{deg } T_2^* = \text{deg } F_{\beta} + \text{deg } F_{\beta+1} + \text{deg } D_{\beta} - \text{deg } F_{\beta-j}^{q^j} + \text{deg } A_j \]

\[ = r \text{deg } F_{\beta} - \text{deg } F_{\beta+1} - \text{deg } F_{\beta-j}^{q^j} + a^* \]

\[ = r^\beta - (\beta + 1)q^{\beta+1} - (\beta - j)q^\beta + a^* \]

Therefore \( \beta + d < (\beta + 1)d \). Because \( \beta - j \geq \beta - d \rightarrow - (\beta - j) \geq - (\beta - d) \) we have

\[ r^\beta q^\beta - (\beta + 1)q^{\beta+1} - (\beta - d)q^\beta < 0. \]

Because \( 0 \leq j \leq d \), \( \beta - j \geq \beta - d \rightarrow - (\beta - j) \geq - (\beta - d) \) we have

\[ r^\beta q^\beta - (\beta + 1)q^{\beta+1} - (\beta - d)q^\beta \rightarrow - \infty. \]

Therefore we may chose \( \beta \) so large that every term of \( T_2 \) is negative.

That is \( |T_2| = q^{\text{deg } T_2} < q^0 = 1 \rightarrow |T_2| < 1. \)

3) We will prove the claim of the theorem. By the definition of \( T_1 \) and \( T_2 \), we can write

\[ T_1 + T_2 = F_{\beta} P(\xi) Q(\xi). \]

Hence we obtain

\[ |T_1 + T_2| = |F_{\beta}| |P(\xi)||Q(\xi)|. \quad (17) \]

Since \( |T_1| \geq 1 \) and \( |T_2| < 1 \), we get

\[ |T_1 + T_2| = \max (|T_1|, |T_2|) = |T_1|. \quad (18) \]

By (17) and (18), we obtain

\[ |P(\xi)||Q(\xi)| = |T_1||F_{\beta}|^{-1}. \quad (19) \]

Let \( |\xi| = q^\lambda \). By (1) and since

\[ |r^{q_k} F_{k-1}^{-1}| = q^{\deg (r^{q_k} F_{k-1}^{-1})} = q^{\deg r^{q_k} - \deg F_k} = q^{0 - k q^k} = q^{-k q^k}. \]

we get \( |\xi| = q^{-k q^k} = q^{-\lambda q^0} = q^0. \) Therefore \( \lambda = 0 \). Since \( \max (a, \lambda) = \max (a, 0) = a \) and by Lemma 2, we find

\[ |Q(\xi)| \leq q^{a (q^d - d + 1) + (q^d - d) \max (a, \lambda)} \]

\[ \leq q^{a q^d + a q^d} \]
\[ |Q(\xi)| = q^{a(d+1)q^d}. \quad (20) \]

Since \( h = h(P) = q^a \),
\[ a = \frac{\log h}{\log q}. \quad (21) \]

By (6) and (21) we find
\[ adq^d > \frac{kq^k}{q}. \quad (22) \]

Consider the sequence
\[ \{ q^{-1}, q^0, q^1, q^2, \ldots \} \]

There are \( \beta \) non-negative integers such that
\[ \beta q^{\beta - 1} \leq \frac{adq^d}{kq^k} < \beta q^\beta. \quad (23) \]

Because, by (22)
\[ \frac{1}{q} \leq \frac{adq^d}{kq^k}. \]

From (21) we obtain the following statement for the above \( \beta \)
\[ \frac{adq^d}{kq^k} < q^\beta \leq \frac{adq^{d+1}}{kq^k}. \quad (24) \]

Further, by (24)
\[ |F_\beta| = q^{d+\beta} = q^\beta q^\beta. \quad (25) \]

By (19),(20),(23),(25) and since \(|T_1| \geq 1\) we get
\[ |P(\xi)| = |T_1||F_\beta|^{-1}|Q(\xi)|^{-1} \]
\[ \geq |F_\beta|^{-1}|Q(\xi)|^{-1} \]
\[ \geq q^{-\frac{1}{k}(adq^{d-k+1})-a(d+1)q^d} \]
\[ = q^{-aq^d(\frac{1}{k}-d-1)}. \quad (26) \]

By (26) and since \( h = q^a \) we have
\[ |P(\xi)| \geq h^{-aq^d(\frac{1}{k}-d-1)}. \]

This is the claim of the theorem 1.
Proof of the Theorem 2:

Let the degree of the polynomial $P$ in Theorem 1 be $\vartheta(P) = d \leq n$ and let its height be $h(P) = h \leq H$. By (4),

$$|P(\xi)| \geq H^{-aq^d\left(\frac{q^{1-k}}{k} - 1\right)}.$$  \hspace{1cm} (27)

(27) and (6) and by the definition of Mahler’s classification

$$\Gamma_n(H, \xi) \geq H^{-aq^d\left(\frac{q^{1-k}}{k} - 1\right)}$$

for all sufficiently large natural numbers $n$ and $H$. Hence consequently

$$\log \Gamma_n(H, \xi) \geq \left[ -aq^d\left(\frac{q^{1-k}}{k} - 1\right) \right] \log H$$

$$\frac{-\log \Gamma_n(H, \xi)}{\log H} \leq aq^d\left(\frac{q^{1-k}}{k} - 1\right)$$ \hspace{1cm} (28)

$$\Gamma_n(\xi) = \lim_{H \to \infty} \sup \frac{-\log \Gamma_n(H, \xi)}{\log H} \leq aq^d\left(\frac{q^{1-k}}{k} - 1\right).$$ \hspace{1cm} (29)

That is, for every index $n$

$$\Gamma_n(\xi) < \infty.$$  

By the definition of Mahler’s classification, $\mu(\xi) = \infty$. This shows $\xi$ can never belong to the class $U$ so that it belongs to the class $S$ or to the class $T$. 

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