SOME FIXED POINT THEOREMS IN HILBERT SPACES

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Abstract. Some general fixed points theorems in Hilbert spaces are proved which generalize the results from [1].

AMS Subject classification (1991): 24 H 25
Key words and phrases: fixed point, Hilbert spaces.

1. INTRODUCTION

Let $R_+$ be denote the set of all non-negative reals. Let $H$ the set of all real function $g(t_1, ..., t_5): R^5_+ \rightarrow R_+$ satisfying the following conditions:

(H1): $g$ is non-decreasing in variables $t_4$ and $t_5$,
(H2): $g(u,0,0,u,u) < u$, $\forall u > 0$,
(H3): there exists $h \in (0, 1)$ such that for every $u, v \in R_+$ with
(Ha): $u \leq g(v,u,u+u,0)$, or
(Hb): $u \leq g(v,u,v,0,u+v)$,
we have $u \leq h.v$.

Ex. 1. $g(t_1, ..., t_5) = q.\max\{t_1, t_2, t_3, t_4 + t_5\}$ where $q \in (0, 1)$ and $h = q$.

(H1). Obviously.
(H2). $g(u,0,0,u,u) = q.u < 0$, $\forall u > 0$.

(Ha). Let $u, v \geq 0$ be such that $u \leq g(v,u,u,v,0)$ then $u \leq q.\max\{v, u, \frac{1}{2}(u+v), 0\}$ which implies $u \leq q.v = h.v$.
(Hb). If $u \leq g(v,u,v,0,u+v)$, similarly, we have $u \leq h.v$.

Ex. 2. $g(t_1, ..., t_5) = [at_1^k + bt_2^k + ct_3^k + d(t_4t_5)^{\frac{k}{2}}]^{\frac{1}{k}}$ where $k \geq 1; \ a > 0; \ b, c, d \geq 0$ and $a + b + c + d < 1$.

(H1). Obviously.
(H2). $g(u,0,0,u,u) = [au^k + du^k]^{\frac{1}{k}} = (a + d)^{\frac{1}{k}}.u < u$, $\forall u > 0$. 
(H₃). Let \( u, v \in R_+ \) be such that \( u \leq g(v, v, u, u + v, 0) \), then we have
\[
u = h_1, v \quad \text{with} \quad h_1 = \left( \frac{a + b}{1 - c} \right)^{\frac{1}{k}} < 1.
\]

If \( u \leq g(v, u, v, 0, u + v) \), similarly, we have \( u \leq h_2, v \) with \( h_2 = \left( \frac{a + c}{1 - b} \right)^{\frac{1}{k}} < 1. \)

Thus \( g \) satisfies condition (H₃) with \( h = \max \{h_1, h_2\} \).

Ex. 3. \( g(t_1, \ldots, t_3) = [a t_1^2 + b t_2^2 + c t_3^2 + d t_4 t_5]^{\frac{1}{2}} \)

where \( a > 0, \ b, c, d \geq 0 \) and \( a + b + c + d < 1. \)

(H₁). Obviously.

(H₂). \( g(u, 0, 0, u, u) = [a u^2 + d u^2]^{\frac{1}{2}} \leq (a + d)^{\frac{1}{2}} u < u, \ \forall u > 0. \)

(H₃). Let \( u, v \in R_+ \) be such that \( u \leq g(v, v, u, u + v, 0) \), then we have
\[
u = h_1, v \quad \text{with} \quad h_1 = \left( \frac{a + c + a v^2}{1 - b + v^2 (1 - b - c)} \right)^{\frac{1}{k}} < 1.
\]

If \( u \leq g(v, u, v, 0, u + v) \), similarly, we have \( u \leq h_2, v \) where \( h_2 \in (0, 1) \).

Thus \( g \) satisfies condition (H₃) with \( h = \max \{h_1, h_2\} \).

Remark 1. There exists the functions \( g : R^3_+ \rightarrow R_+ \) which satisfies conditions
(H₁) - (H₃) and is decreasing in variable \( t_2 \) and \( t_3 \).

Ex. 4. \( g(t_1, \ldots, t_3) = [a t_1^2 + \frac{b t_4 t_5}{t_2^2 + t_3^2 + 1}]^{\frac{1}{2}} \)

where \( a > 0, \ b \geq 0 \) and \( a + b < 1. \)

(H₁). Obviously.

(H₂). \( g(u, 0, 0, u, u) = (a + b)^{\frac{1}{2}} u < u, \ \forall u > 0. \)

(H₃). Let \( u, v \in R_+ \) be such that \( u \leq g(v, v, u, u + v, 0) \), then we have \( u \leq a^{\frac{1}{2}} v = h, v \),

where \( h \in (0, 1) \). If \( u \leq g(v, u, v, o, u + v) \), then \( u \leq hv \) where \( h = a^{\frac{1}{2}} < 1. \)

2. MAIN RESULTS

Theorem 1. Let \( T_1 \) and \( T_2 \) be two mappings from Hilbert space \( X \) into itself such inequality
\[
\|T_1 x - T_2 y\| \leq g(\|x - y\|, \|x - T_1 x\|, \|y - T_2 y\|, \|x - T_2 y\|, \|y - T_1 x\|)
\]
holds for all \( x, y \in X \)

where \( g \in H \), then \( F_{T_1} = F_{T_2} \), where \( F_T = \{x \in X : \ x = T x\}. \)
**Proof.** Let \( u \in F_{T_2} \) be then
\[
\|u - T_2u\| = \|T_1u - T_2u\| \leq g\left(\|u - u\|, \|u - T_1u\|, \|u - T_2u\|, \|u - T_1u\|\right) = \\
= g(0,0,\|u - T_2u\|,\|u - T_2u\|,0).
\]
By (H_a) we have \( \|u - T_2u\| \leq 0 \) which implies \( u = T_2u \) thus \( u \in F_{T_2} \) and \( F_{T_2} \subseteq F_{T_2} \). Similarly, by (H_b), we have \( F_{T_2} \subseteq F_{T_2} \).

**Theorem 2.** Let \( T_1 \) and \( T_2 \) be two mappings from Hilbert space \( X \) into itself such that inequality (1) holds for all \( x, y \in X \) where \( g \) satisfies (H_2). If \( T_1 \) and \( T_2 \) have a common fixed point \( z \), then \( z \) is a unique common fixed point for \( T_1 \) and \( T_2 \).

**Proof.** Suppose that \( T_1 \) and \( T_2 \) have a second common fixed point \( z' \neq z \). Then
\[
\|z - z'\| = \|T_1z - T_2z'\| \leq g\left(\|z - z\|, \|z - T_1z\|, \|z' - T_2z'\|, \|z - T_2z'\|, \|z' - T_1z\|\right) = \\
= g(\|z - z\|,0,0,\|z - z\|,\|z - z\|) < \|z - z'\|, 
\]
a contradiction.
In [1] is proved following theorem.

**Theorem 3.** Let \( X \) be a closed subset of a Hilbert space and \( T_1 \) and \( T_2 \) be mappings of \( X \) into itself satisfying
\[
(2) \quad \|T_1x - T_2y\|^2 \leq a\|x - y\|^2 + b\|y - T_2y\|^2 + c\|x - T_1x\|^2 \frac{1 + \|x - T_1x\|^2}{1 + \|x - y\|^2} 
\]
for all \( x, y \) in \( X \), where \( a, b, c \) are non-negative reals with \( a + b + c < 1 \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point in \( X \).

The purpose of this paper is to extend Theorem 3 and others results from [1] for the functions \( g \in H \).

**Theorem 4.** Let \( X \) be a closed subset of a Hilbert space and \( T_1 \) and \( T_2 \) be mappings of \( X \) into itself satisfying inequality (1) for all \( x, y \) in \( X \), where \( g \in H \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point in \( X \).

**Proof.** For arbitrary \( x_0 \in X \), define the sequence \( \{x_n\} \) as
\[
x_1 = T_1x_0, \quad x_2 = T_2x_1, \quad \ldots, \quad x_{2n} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}, \ldots 
\]
Then we have
\[
\|x_{2n+1} - x_n\| = \|T_1x_{2n} - T_2x_{2n-1}\| \leq g\left(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - T_1x_{2n}\|, \|x_{2n-1} - T_2x_{2n-1}\|\right), \\
\|x_{2n} - T_1x_{2n-1}\| \leq g\left(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - x_{2n-1}\|, \|x_{2n-1} - x_{2n}\|\right), \\
\|x_{2n-1} - x_{2n+1}\| \leq g\left(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - x_{2n-1}\|, \|x_{2n-1} - x_{2n}\|\right)
\]
which implies, by condition (H_b), that
\[
\|x_{2n+1} - x_n\| \leq h\|x_{2n-1} - x_{2n}\|.
\]
Similarly, by condition (H_b), we have
\[
\|x_{2n} - x_{2n-1}\| \leq h\|x_{2n-1} - x_{2n-2}\|.
\]
Hence we get
\[
\|x_{n+1} - x_n\| \leq h^n\|x_1 - x_0\| \quad \text{for all} \quad n \in \mathbb{N}.
\]
Hence \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is closed, there exists \( u \in X \) which is the limit of \( x_n \), i.e. \( \lim x_n = u \). Since \( x_{2n+1} = T_1 x_{2n} \) and \( x_{2n+2} = T_2 x_{2n+1} \) are subsequences of \( \{x_n\} \), \( \{T_1 x_{2n}\} \) and \( \{T_2 x_{2n+1}\} \) also converge to the same limit \( u \). We now prove that \( u \) is a common fixed point of \( T_1 \) and \( T_2 \). Consider

\[
\|u - T_2 u\|^2 = \|(u - x_{2n+1}) + (x_{2n+1} - T_2 u)\|^2 \leq \|u - x_{2n+1}\|^2 + 2 \Re \langle u - x_{2n+1}, x_{2n+1} - T_2 u \rangle + \\
+ \|T_2 u - u\| \|x_{2n+1} - T_2 u\|.
\]

Letting \( n \to \infty \), so that \( x_{2n}, x_{2n+1} \to u \) and \( \Re \langle u - x_{2n+1}, x_{2n+1} - T_2 u \rangle \to 0 \) we get

\[
\|u - T_2 u\| \leq g(0,0,\|T_2 u - u\|,\|T_2 u - u\|,0).
\]

By condition (Hb) follows that \( \|u - T_2 u\| \leq 0 \), which implies \( T_2 u = u \). By Theorems 1 and 2 follows that \( u \) is unique common fixed point for \( T_1 \) and \( T_2 \).

**Corollary 1.** Let \( X \) be a closed subset of a Hilbert space and \( T_1 \) and \( T_2 \) be mappings on \( X \) into itself such that

a) \( \|T_1 x - T_2 y\| \leq k \cdot \max\{\|x - y\|,\|x - T_1 x\|,\|y - T_2 y\|,\|x - T_2 y\| + \|y - T_1 x\|\} \)

where \( k \in (0, 1) \), or

b) \( \|T_1 x - T_2 y\|^k \leq a\|x - y\|^k + b\|x - T_1 x\|^k + c\|y - T_2 y\|^k + d\|x - T_2 y\| \cdot \|y - T_1 x\|^{k/2} \)

where \( k \geq 1 \), \( a > 0 \), \( b, c, d \geq 0 \) and \( a + b + c + d < 1 \), or

c) \( \|T_1 x - T_2 y\|^2 \leq a\|x - y\|^2 + b\|y - T_2 y\|^2 \frac{1 + \|x - T_1 x\|^2}{1 + \|x - y\|^2} + \\
+ c\|x - T_1 x\|^2 \frac{1 + \|y - T_2 y\|^2}{1 + \|x - y\|^2} + d\|x - T_2 y\| \cdot \|y - T_1 x\| \frac{1 + \|x - y\|^2}{1 + \|x - y\|^2} \)

where \( a > 0 \), \( b, c, d \geq 0 \) and \( a + b + c + d < 1 \), holds for all \( x, y \) in \( X \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point in \( X \).

**Remark 2.** From Corollary 1(c) for \( d = 0 \) follows Theorem 3.

**Theorem 5.** Let \( X \) be a closed subset of a Hilbert space and \( \{T_n\}_{n \in N} \) a sequence of mapping on \( X \) into itself satisfying inequality

\[(3) \quad \|T_n x - T_{n+1} y\| \leq g(\|x - y\|,\|x - T_n x\|,\|y - T_{n+1} y\|,\|x - T_{n+1} y\|,\|y - T_n x\|) \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad g \in \mathcal{H}. \]

Then the sequence \( \{T_n\}_{n \in N} \) has unique common point in \( X \).

**Proof.** By Theorem 4, \( T_1 \) and \( T_2 \) have a unique common fixed point. By Theorem 1, \( z \) is unique fixed point for the sequence \( \{T_n\}_{n \in N} \).
Corollary 2. Let $X$ be a closed subset of a Hilbert space and $\{T_n\}_{n \in \mathbb{N}}$ a sequence of mappings on $X$ into itself such that

a) $\|T_n x - T_{n+1} x\| \leq k \max\{\|x - y\|, \|x - T_n x\|, \|y - T_{n+1} x\|\}$

where $k \in (0, 1)$, or

b) $\|T_n x - T_{n+1} y\| \leq a \|x - y\| + b \|x - T_n x\| + c \|y - T_{n+1} y\| + d \|\|x - T_{n+1} y\| - \|y - T_n x\|\|^{1/2}$

where $k \geq 1$, $a > 0$, $b, c, d \geq 0$ and $a + b + c + d < 1$, or

c) $\|T_n x - T_{n+1} y\| \leq a \|x - y\| + b \|y - T_{n+1} y\| \frac{1 + \|x - T_n x\|^2}{1 + \|x - y\|^2} + c \|x - T_n x\| \frac{1 + \|y - T_{n+1} y\|^2}{1 + \|x - y\|^2} + d \|\|x - T_{n+1} y\| - \|y - T_n x\|\|^{1/2}$

where $a > 0$, $b, c, d \geq 0$ and $a + b + c + d < 1$,

holds for all $x, y$ in $X$. Then the sequence $\{T_n\}_{n \in \mathbb{N}}$ have a unique common fixed point.

Theorem 6. Let $X$ be a closed subset of a Hilbert space and $T_1$ and $T_2$ be mappings on $X$ into itself satisfying inequality

$\|T_1^p x - T_2^q y\| \leq g(\|x - y\|, \|x - T_1^p x\|, \|y - T_2^q y\|, \|x - T_2 q y\|, \|y - T_1^p x\|)$

for all $x, y \in X$, where $g \in H$, and $p, q$ are some positive integers. Then $T_1$ and $T_2$ have a unique common point in $X$.

Proof. $T_1^p$ and $T_2^q$ satisfy all conditions of the Theorem 4. Hence they have a unique common fixed point, say $u$, so that $T_1^p u = u$, $T_2^q u = u$.

Now, $T_1^p u = u$ implies $T_1^p(T_1^p u) = T_1 u$ and $T_1^p(T_2 u) = T_1 u$. Hence $T_1 u$ is a fixed point of $T_1^p$. Similarly, $T_2 u$ is a fixed point of $T_2^q$. Now if $u \neq T_2 u$, we have

$\|u - T_2 u\| = \|T_1^p u - T_2^q(T_2 u)\| \leq g(\|u - T_2 u\|, \|u - T_1^p u\|, \|T_2 u - T_2^q(T_2 u)\|, \|u - T_2^q(T_2 u)\|, \|u - T_2^q(T_2 u)\|, \|u - T_2 u\|)$

which is a contradiction. Thus $u = T_2 u$. Similarly we get $u = T_1 u$. If $v$ is another common fixed point of $T_1$ and $T_2$ then clearly $v$ is also a common fixed point of $T_1^p$ and $T_2^q$. By Theorem 4, $T_1^p$ and $T_2^q$ have a unique common fixed point.

Corollary 3. Let $X$ be a closed subset of a Hilbert space and $T_1$ and $T_2$ be mappings on $X$ into itself such that

a) $\|T_1^p x - T_2^q y\| \leq k \max\{\|x - y\|, \|x - T_1^p x\|, \|y - T_2^q y\|, \|y - T_1^p x\|\}$

where $k \in (0, 1)$, or
where \( k \geq 1, \ a > 0, \ b, c, d \geq 0 \) and \( a + b + c + d < 1 \), or

\[
\| T_1^{p} x - T_2^{q} y \|^{k} \leq a\| x - y \|^{k} + b\| x - T_1^{p} x \|^{k} + c\| y - T_2^{q} y \|^{k} + d\| x - T_1^{p} x \| \cdot \| y - T_1^{p} x \|^{\frac{k}{2}}
\]

holds for all \( x, y \) in \( X \) and \( p, q \) positive integers. Then \( T_1 \) and \( T_2 \) have a unique common fixed point.

**Remark 3.** Corollary 3(c) for \( d = 0 \) is Corollary 2 of [1].

**REFERENCES**