GENERAL MIXED MULTIVALUED MILDLY NONLINEAR VARIATIONAL INEQUALITIES IN H-SPACES

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Abstract - A general mixed multivalued mildly nonlinear variational inequality is considered. We establish an existence for (GMMMNVI) by replacing convexity assumptions with merely topological properties.

1. INTRODUCTION

In the years 1983-1985, Horvath [2] obtained minimax inequalities by replacing convexity assumptions with merely topological properties.

In this paper, we consider a general mixed multivalued mildly nonlinear variational inequality, introduced and studied by Siddiqi et al [3], and prove the existence of its solution without convexity.

Let $X$ and $Y$ be two real Banach spaces. Let $K \subset X$ be a nonempty closed convex subset of $X$, $g: K \to K$ be a continuous mapping, $T, A: K \to 2^{L(X,Y)}$ be the multivalued mappings, where $L(X,Y)$ is the space of all linear continuous mappings from $X$ into $Y$.

Consider the general mixed multivalued mildly nonlinear variational inequality (GMMMNVI). Find $u \in K$, $x \in T(u)$ and $y \in A(u)$ such that $g(u) \in K$ and

$$< x - y, g(v) - g(u) > + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } g(v) \in K, \quad \ldots \ (1.1)$$

where $b: K \times K \to R$ is a nonlinear form.

2. EXISTENCE THEORY

First, we give definitions for the existence theorem for (GMMMNVI), that can be found in [1].

Definition 2.1. Let $X$ be a topological space and $\{\Gamma_\lambda\}$ be a family of nonempty contractible subsets of $X$, indexed by the finite subsets of $X$. 
A subset $D \subseteq X$ is called weakly H-convex if, for every finite subset $A \subseteq D$, it results that $\bigcap_{A} \Gamma_{A} \cap D$ is nonempty and contractible. This is equivalent to saying that the pair $(D, \{ \Gamma_{A} \cap D \})$ is an H-space.

A subset $K \subseteq X$ is called H-compact if there exists a compact, weakly H-convex set $D \subseteq X$ such that $K \cup A \subseteq D$ for every finite subset $A \subseteq X$.

A multifunction $G : X \to 2^X$, is called H-KKM mapping if

$$\Gamma_{A} \subseteq \bigcup_{x \in A} G(x),$$

for every finite subset $A \subseteq X$.

**Lemma 2.1.** [1]. Let $(X, \{ \Gamma_{A} \})$ be an H-space and $F : X \to 2^X$, an H-KKM multifunction such that

(a) For each $x \in X$, $F(x)$ is compactly closed, that is $B \cap F(x)$ is closed in $B$, for every compact set $B \subseteq X$;

(b) There are a compact set $L \subseteq X$ and an H-compact set $K \subseteq X$ such that, for each weakly H-convex set $D$ with $K \subseteq D \subseteq X$, we have

$$\bigcap_{x \in D} \{ F(x) \cap D \} \subseteq L$$

Then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

Now, we state and prove the main Theorem of this paper.

**Theorem 2.1.** Let $(X, \{ \Gamma_{k} \})$ be an H-Banach space and $Y$ be an ordered Banach space. Assume that

1°. $T, A : X \to 2^{L(X,Y)}$ be the compact valued, continuous multivalued mappings;

2°. $g : X \to X$ is a continuous mapping;

3°. $b : X \times X \to R$ is a continuous, nonlinear form;

4°. For each $v \in X$, $B_v = \{ u \in X : \exists x \in T(v), y \in A(v) \text{ such that } x-y, g(v)-g(u) > b(u, g(v)) - b(u, g(u)) < 0 \}$ is H-convex or empty;

5°. There exists a compact set $L \subseteq X$ and an H-compact set $E \subseteq X$ such that for every weakly H-convex set $D$ with $E \subseteq D \subseteq X$

$\{ v \in D : \exists x \in T(v), y \in A(v) \text{ such that } x-y, g(v)-g(u) > b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } u \in D \} \subseteq L$.

Then (GMMMNV) is solvable.
Proof: Let

\[ F(u) = \left\{ v \in X : \exists x \in T(v), y \in A(v) \text{ such that} \right. \]
\[ \left. \quad \langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) \geq 0 \right\}, \quad u \in X \]

First we prove that \( F \) is an H-KKM mapping and the conditions (a) and (b) of Lemma 2.1 hold.

Suppose that \( F \) is not an H-KKM mapping. Then there exists a finite subset \( K \subseteq X \)
such that \( \Gamma_k \not\subset \bigcup_{u \in K} F(u) \). Thus there exists \( z \in \Gamma_k \) such that

\[ z \notin F(u), \text{ for all } u \in K, \]

that is

\[ \langle x - y, g(z) - g(u) \rangle + b(u, g(z)) - b(u, g(u)) < 0, \text{ for all } u \in K, x \in T(z), y \in A(z). \]

By assumption 4\( ^9 \), \( K \subseteq B_z \) and \( \Gamma_k \subseteq B_z \), since \( B_z \) is H-convex. Therefore \( z \in B_z \), that is there exists \( x \in T(z), y \in A(z) \) such that

\[ \langle x - y, g(z) - g(z) \rangle + b(z, g(z)) - b(z, g(z)) < 0, \]

which is not possible. Thus \( \Gamma_k \subset \bigcup_{u \in K} F(u) \) for every finite subset \( K \) of \( X \), so that \( F \) is an H-KKM mapping.

Next, we prove that for every \( u \in X \), \( F(u) \) is closed. Indeed, let \( \{v_n\} \) be a sequence in \( F(u) \) such that \( v_n \to v \in X \), \( g(v_n) \to g(v) \in X \). Since \( v_n \in F(u) \) for all \( n \), there exists \( x_n \in T(v) \)

and \( y_n \in A(v) \) such that

\[ \langle x_n - y_n, g(v_n) - g(u) \rangle + b(u, g(v_n)) - b(u, g(u)) \geq 0. \]

Since \( T(v) \) and \( A(v) \) are compact, without loss of generality, we can assume that there exists \( x_0 \in T(v) \) and \( y_0 \in A(v) \) such that \( x_n \to x_0 \) and \( y_n \to y_0 \). Now since \( \langle \cdot, \cdot \rangle \), \( b(\cdot) \) are continuous and \( x_n \to x_0 \), \( y_n \to y_0 \), \( v_n \to v \), \( g(v_n) \to g(v) \), we have

\[ \langle x_n - y_n, g(v_n) - g(u) \rangle + b(u, g(v_n)) - b(u, g(u)) \to \langle x_0 - y_0, g(v_0) - g(u) \rangle + b(u, g(v_0)) - b(u, g(u)) \geq 0. \]

Therefore \( v_0 \in F(u) \) and so \( F(u) \) is closed for every \( u \in X \), that is, the condition (a) of the Lemma 2.1 holds. It is easy to see that the assumption 5\( ^9 \) of the Theorem 2.1 is same one with the condition (b) of the Lemma 2.1.

Then by Lemma 2.1

\[ \bigcap_{u \in X} F(u) \neq \emptyset, \]

consequently, there exists \( u_0 \in X \), \( x_0 \in T(u_0) \), \( y_0 \in A(u_0) \) such that \( g(u_0) \in X \) such that
\[ <x_0 - y_0, g(v) - g(u_0)> + b(u_0, g(v)) - b(u_0, g(u_0)) \geq 0, \] for all \( g(v) \in X \).

This completes the proof.

**REFERENCES**

