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PROXIMAL METHODS FOR GENERALIZED NONLINEAR QUASI-VARIATIONAL INCLUSIONS

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Abstract- We consider the solvability, based on iterative algorithms, of the generalized nonlinear quasi-variational inclusion problems involving the relaxed Lipschitz and relaxed monotone mappings.
Key words- Generalized nonlinear quasi-variational inclusion problems, Relaxed Lipschitz and relaxed monotone mappings, Hausdorff metric, Algorithm.

1. INTRODUCTION

Variational inequalities theory, which was introduced by Stampacchia [14] in 1964, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, physical, regional, social, pure and applied sciences (c.f. [5, 6, 8-10, 12] and references therein). This field is dynamic and is experiencing an explosive growth in both theory and applications, as a consequence, research techniques and problems are drawn from various fields. An important and useful generalization of variational inequalities is called variational inclusion, which was introduced and studied by Hassouni and Moudafi [7].

In this paper we study a class of generalized nonlinear quasi-variational inclusions with noncompact valued mappings and propose a proximal point algorithm for computing its approximate solution. We prove that the approximate solution obtained by proposed algorithm converges to the exact solution of our inclusion. Some special cases are also discussed.

2. PRELIMINARIES

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle .,. \rangle$, respectively. We denote by 2^H the family of all nonempty subsets of H. Given multivalued mappings $A, S, T: H \to 2^H$, single-valued mappings $g, m : H \to H$ and a bifunction $N : H\times H \to H$. We consider the following problem of finding $u\in H$, $x\in Su$, $y\in Tu$ and $z\in Au$ such that

$$g(u) - m(z) \in \operatorname{dom} \partial\phi(.,u)$$

$$\langle N(x,y), v - g(u) \rangle \geq \phi(g(u) - m(z), u) - \phi(v, u), \text{ for all } v\in H, \quad (2.1)$$

where $\phi : H\times H \to R \cup \{+\infty\}$ be such that for each fixed $v\in H$, $\phi(.,v) : H\times H \to R \cup\{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\phi$ denotes the subdifferential function of ϕ.

The problem (2.1) is called generalized nonlinear quasi-variational inclusion problem (GNQVIP).

Special cases- If $\phi(u,v) = \phi(u)$, for all $v\in H$, then problem (2.1) reduces to the following generalized set-valued nonlinear quasivariational inclusion problem (GSVNQVIP) considered by Shim et al [13].

(GSVNQVIP): Find $u \in H$, $x \in Su$, $y \in Tu$ and $z \in Au$ such that

$$g(u) - m(z) \in \text{dom } \partial\phi \text{ and}$$

$$\langle N(x,y), v - g(u) \rangle \geq \phi(g(u) - m(z)) - \phi(v), \text{ for all } v \in H. \qquad (2.2)$$

In order to prove our main result, we need the following concepts and results.

Definition 2.1 [2]- Let H be a Hilbert space and $G:H \to 2^H$ a maximal monotone mapping. For any fixed $\eta > 0$, the mapping $J^G_\eta : H \to H$ defined by

$$J^G_\eta(u) = (I + \eta G)^{-1}(u), \text{ for all } u \in H,$$

is called the resolvent operator of G, where I stands for the identity mapping on H.

Lemma 2.1 [1]- Let X be a reflexive Banach space endowed with a strictly convex norm and $\phi : X \to R \cup \{+\infty\}$ a proper convex lower semicontinuous function. Then $\partial\phi : X \to 2^{X^*}$ is a maximal monotone mapping.

Lemma 2.2 [3]- Let $G:H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J^G_\eta : H \to H$ of G is nonexpansive, that is for all $u, v \in H$

$$\| J^G_\eta(u) - J^G_\eta(v) \| \leq \| u - v \|.$$

Definition 2.2- A mapping $g : H \to H$ is said to be

(i) α - strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle g(u) - g(v), u - v \rangle \geq \alpha \| u - v \|^2, \text{ for all } u, v \in H;$$

(ii) β - Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\| g(u) - g(v) \| \leq \beta \| u - v \|, \text{ for all } u, v \in H.$$

Definition 2.3 [15]- Let $S:H \to 2^H$ be a mapping. An operator $N(.,.):H\times H \to H$ is said to be relaxed Lipschitz with respect to S in the first argument if there exists a constant $\kappa \geq 0$ such that

$$\langle N(x_1,.) - N(x_2,.), u - v \rangle \leq \kappa \| u - v \|^2 \text{ for all } x_1 \in Su, x_2 \in Sv \text{ and } u,v \in H.$$

Definition 2.4 [15]- Let $T:H \to 2^H$ be a mapping. An operator $N(.,.):H\times H \to H$ is said to be relaxed monotone with respect to T in the second argument if there exists a constant $c > 0$ such that

$$\langle N(.,y_1) - N(.,y_2), u - v \rangle \geq c \| u - v \|^2, \text{ for all } y_1 \in Tu, y_2 \in Tv \text{ and } u,v \in H.$$

Defination 2.5 [4]- A mapping $S:H \to 2^H$ is said to be Lipschitz continuous if there exists a constant $\upsilon > 0$ such that

$$\| x_1 - x_2 \| \leq M(Su_1, Su_2) \leq \upsilon \| u_1 - u_2 \|, \text{ for all } x_i \in Su_i \text{ and } u_i \in H, i = 1, 2,$$

where $M(.,.)$ is Hausdorff metric on H.

3. ITERATIVE ALGORITHM

The following Lemma 3.1 ensures that (2.1) is equivalent to a fixed point problem.

Lemma 3.1- The set of elements (u, x, y, z) is a solution of the problem (2.1) if and only if (u, x, y, z) satisfies the relation :

$$g(u) = m(z) + J^{\partial\phi(.,u)}_\eta [g(u) - \eta N(x,y) - m(z)], \qquad (3.1)$$

where $\eta > 0$ is a constant, $J^{\partial\phi(.,u)}_\eta = (I + \eta\, \partial\phi(.,u))^{-1}$ is the resolvent operator of $\partial\phi(.,u)$ and I stands for the identity mapping on H.

Based on Lemma 3.1, we see that generalized nonlinear quasi-variational inclusion problem (2.1) is equivalent to the fixed point problem (3.1). The equation (3.1) can be written as

$$u = (1-\lambda)u + \lambda\,[u - g(u) + m(z) + J^{\partial\phi(.,\,u)}_{\eta}[\,g(u) - \eta\,N(x,y) - m(z)]], \tag{3.2}$$

where $0 < \lambda < 1$ and $\eta > 0$ are both constants.

This fixed point formulation enables us to suggest the following algorithm.

Algorithm 3.1- Let g,m:H $\to$ H be single-valued mappings, N:H$\times$H $\to$ H a bifunction and A, S, T:H $\to$ CB(H) be multivalued mappings, where CB(H) denotes the family of all nonempty bounded closed subset of H. For given $u_0 \in H$, we take $x_0 \in Su_0$, $y_0 \in Tu_0$, $z_0 \in Au_0$ and for $\eta > 0$, assume

$$u_1 = (1-\lambda)u_0 + \lambda\,[u_0 - g(u_0) + m(z_0) + J^{\partial\phi(.,\,u_0)}_{\eta}[\,g(u_0) - \eta\,N(x_0,y_0) - m(z_0)]\,],$$

where $0 < \lambda < 1$ is a constant.

Since $x_0 \in Su_0 \in CB(H)$, $y_0 \in Tu_0 \in CB(H)$,, $z_0 \in Au_0 \in CB(H)$, by Nadler [11] there exist $x_1 \in Su_1$, $y_1 \in Tu_1$, $z_1 \in Au_1$ such that

$$\|x_1 - x_0\| \le (1+1)\ M(Su_1, Su_0),$$

$$\|y_1 - y_0\| \le (1+1)\ M(Tu_1, Tu_0),$$

$$\|z_1 - z_0\| \le (1+1)\ M(Au_1, Au_0),$$

where M(.,.) is the Hausdorff metric on CB(H). Let

$$u_2 = (1-\lambda)u_1 + \lambda\,[u_1 - g(u_1) + m(z_1) + J^{\partial\phi(.,\,u_1)}_{\eta}[\,g(u_1) - \eta\,N(x_1,y_1) - m(z_1)]],$$

where $0 < \lambda < 1$ is a constant.

Since $x_1 \in Su_1 \in CB(H)$, $y_1 \in Tu_1 \in CB(H)$, $z_1 \in Au_1 \in CB(H)$, there exist $x_2 \in Su_2$, $y_2 \in Tu_2$, $z_2 \in Au_2$ such that

$$\|x_1 - x_2\| \le (1+2^{-1})\ M(Su_1, Su_2),$$

$$\|y_1 - y_2\| \le (1+2^{-1})\ M(Tu_1, Tu_2),$$

$$\|z_1 - z_2\| \le (1+2^{-1})\ M(Au_1, Au_2),$$

By induction, we can obtain sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as

$$u_{n+1} = (1-\lambda)u_n + \lambda\,[u_n - g(u_n) + m(z_n) + J^{\partial\phi(.,\,u_n)}_{\eta}[\,g(u_n) - \eta\,N(x_n,y_n) - m(z_n)]\,], \tag{3.3}$$

$$x_n \in Su_n,\ \ \|x_n - x_{n+1}\| \le (1+(1+n)^{-1})\ M(Su_n, Su_{n+1}),$$

$$y_n \in Tu_n,\ \ \|y_n - y_{n+1}\| \le (1+(1+n)^{-1})\ M(Tu_n, Tu_{n+1}),$$

$$z_n \in Au_n, \; \| z_n - z_{n+1} \| \le (1 + (1+n)^{-1}) \; M(Au_n, Au_{n+1}),$$

$n = 0, 1, 2, \ldots\ldots$, where $0 < \lambda < 1$ and $\eta > 0$ are both constants.

4. MAIN RESULT

In this section, we prove the existence of solution of the problem (2.1) and the convergence of iterative sequences generated by Algorithm 3.1.

Theorem 4.1- Let $g:H \to H$ be a strongly monotone and Lipschitz continuous mapping with constants $\alpha > 0$, $\beta > 0$, respectively and $m:H \to H$ a Lipschitz continuous mapping with constant $\gamma > 0$. Let $A, S, T:H \to CB(H)$ be Lipschitz continuous mappings with constants $\sigma > 0$, $\xi > 0$ and $\rho > 0$, respectively. Let the bifunction $N:H\times H \to H$ be relaxed Lipschitz continuous with respect to S in first argument with constant $\kappa \le 0$, and relaxed monotone with respect to T in second argument with constant $c > 0$. Let the bifunction $N(.,.)$ be Lipschitz continuous in first and second argument with constants $\delta > 0$ and $\omega > 0$. Let $\phi:H\times H \to R \cup \{+\infty\}$ be such that for each fixed $v \in H$, $\phi(.,v)$ is a proper convex lower semicontinuous function on H. For each $u, v, w \in H$ and $\eta > 0$, let

$$\| J^{\partial\phi(.,u)}_{\eta}(w) - J^{\partial\phi(.,v)}_{\eta}(w) \| \le \mu \| u - v \| \text{ and}$$

if

$$|\eta - (\kappa - c)(\delta\xi + \omega\rho)^{-2}| < [(\kappa - c)^2 - q(2-q)(\delta\xi + \omega\rho)^2]^{1/2} (\delta\xi + \omega\rho)^{-2},$$

$$(\kappa - c) > (\delta\xi + \omega\rho)[q(2-q)]^{1/2},$$

$$q = 2[1 - 2\alpha + \beta^2]^{1/2} + 2\gamma\sigma + \mu, \; q < 1, \tag{4.1}$$

then there exist $u \in H$, $x \in Su$, $y \in Tu$ and $z \in Au$ satisfying the problem (2.1). Moreover,

$$u_n \to u, \; x_n \to x, \; y_n \to y, \; z_n \to z \quad \text{as } n \to \infty,$$

where the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined in Algorithm 3.1.

Proof- From Algorithm 3.1, we have

$$\| u_{n+1} - u_n \| = \| (1-\lambda)u_n - (1-\lambda)u_{n-1} + \lambda[u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) + m(z_n) - m(z_{n-1})$$
$$+ J^{\partial\phi(.,u_n)}_{\eta}[g(u_n) - \eta N(x_n, y_n) - m(z_n)] - J^{\partial\phi(.,u_{n-1})}_{\eta}[g(u_{n-1}) - \eta N(x_{n-1}, y_{n-1}) - m(z_{n-1})]] \|$$

$$\le (1-\lambda)\| u_n - u_{n-1} \| + \lambda \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \| + \lambda \| m(z_n) - m(z_{n-1}) \|$$

$+ \lambda \| J^{\partial\phi(.,u_n)}_\eta [g(u_n) - \eta N(x_n, y_n) - m(z_n)] - J^{\partial\phi(.,u_n)}_\eta [g(u_{n-1}) - \eta N(x_{n-1}, y_{n-1}) - m(z_{n-1})] \|$

$+ \lambda \| J^{\partial\phi(.,u_n)}_\eta [g(u_{n-1}) - \eta N(x_{n-1}, y_{n-1}) - m(z_{n-1})] - J^{\partial\phi(.,u_{n-1})}_\eta [g(u_{n-1}) - \eta N(x_{n-1}, y_{n-1}) - m(z_{n-1})] \|$

$\leq (1-\lambda) \| u_n - u_{n-1} \| + 2\lambda \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \| + 2\lambda \| m(z_n) - m(z_{n-1}) \|$

$+ \mu\lambda \| u_n - u_{n-1} \| + \lambda \| u_n - u_{n-1} - \eta(N(x_n, y_n) - N(x_{n-1}, y_n) + N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})) \|$

$\leq (1-\lambda) \| u_n - u_{n-1} \| + 2\lambda \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \| + 2\gamma\lambda \| z_n - z_{n-1} \| + \mu\lambda \| u_n - u_{n-1} \|$

$$+ \lambda \| u_n - u_{n-1} - \eta(N(x_n, y_n) - N(x_{n-1}, y_n) + N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})) \| . \quad (4.2)$$

By the Lipschitz continuity and strong monotonicity of g, we obtain

$$\| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \|^2 \leq (1 - 2\alpha + \beta^2) \| u_n - u_{n-1} \|^2 . \quad (4.3)$$

Since A, S, T are M-Lipschitz continuous and N is Lipschitz continuous in first and second argument, we have

$$\| N(x_n, y_n) - N(x_{n-1}, y_n) \| \leq \delta \| x_n - x_{n-1} \| \leq \delta\xi (1 + n^{-1}) \| u_n - u_{n-1} \| , \quad (4.4)$$

$$\| N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}) \| \leq \omega \| y_n - y_{n-1} \| \leq \omega\rho (1 + n^{-1}) \| u_n - u_{n-1} \| , \quad (4.5)$$

and

$$\| z_n - z_{n-1} \| \leq \sigma (1 + n^{-1}) \| u_n - u_{n-1} \| . \quad (4.6)$$

Since N is relaxed Lipschitz continuous with respect to S in first argument, relaxed monotone with respect to T in second argument and (4.4), (4.5), we have

$\| u_n - u_{n-1} - \eta(N(x_n, y_n) - N(x_{n-1}, y_n) + N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})) \|^2 \leq \| u_n - u_{n-1} \|^2$

$- 2\eta \langle N(x_n, y_n) - N(x_{n-1}, y_n), u_n - u_{n-1} \rangle - 2\eta \langle N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}), u_n - u_{n-1} \rangle$

$+ \eta^2 \| N(x_n, y_n) - N(x_{n-1}, y_n) + N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}) \|^2$

$\leq \| u_n - u_{n-1} \|^2 - 2\eta\kappa \| u_n - u_{n-1} \|^2 + 2\eta c \| u_n - u_{n-1} \|^2$

$+ \eta^2 (\delta\xi + \omega\rho)^2 (1 + n^{-1})^2 \| u_n - u_{n-1} \|^2$

$$\leq [1 - 2\eta(\kappa - c) + \eta^2 (\delta\xi + \omega\rho)^2 (1 + n^{-1})^2] \| u_n - u_{n-1} \|^2 . \quad (4.7)$$

From (4.2) - (4.7), it follows that

$$\| u_{n+1} - u_n \| \leq \theta_n \| u_n - u_{n-1} \| , \tag{4.8}$$

where

$$\theta_n = \lambda\, q_n + \lambda\, [\, 1 - 2\eta(\kappa - c) + \eta^2\, (\delta\xi + \omega\rho)^2 (1 + n^{-1})^2\,]^{1/2} + (1 - \lambda)$$

and

$$q_n = 2(1 - 2\,\alpha + \beta^2)^{\,1/2} + \; 2\gamma\sigma(1 + n^{-1}) + \mu\,.$$

Letting

$$\theta = \lambda\, q + \lambda\;\; [\, 1 - 2\eta(\kappa - c) + \eta^2\, (\delta\xi + \omega\rho)^2\;\,]^{1/2}\; + (1 - \lambda).$$

We know that $\theta_n \to \theta$ as $n \to \infty$. It follows from (4.1) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (4.8) implies that $\{u_n\}$ is a Cauchy sequence in H and we can suppose that $u_n \to u \in H$ as $n \to \infty$.

Now we prove that $x_n \to x \in Su$, $y_n \to y \in Tu$ and $z_n \to z \in Au$. In fact, it follows from Algorithm 3.1, that

$$\| x_n - x_{n-1} \| \leq \xi\, (\, 1 + n^{-1})\, \| u_n - u_{n-1} \| ,$$

$$\| y_n - y_{n-1} \| \leq \lambda\, (\, 1 + n^{-1})\, \| u_n - u_{n-1} \| ,$$

$$\| z_n - z_{n-1} \| \leq \sigma\, (\, 1 + n^{-1})\, \| u_n - u_{n-1} \| ,$$

that is $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in H. Let $x_n \to x$, $y_n \to y$, $z_n \to z$ as $n \to \infty$. Further, we have

$$\begin{aligned} d(x\,, Su) &= \inf\{\, \| x - v \| : v \in Su\} \\ &\leq \| x - x_n \| + d(x_n\,, Su) \\ &\leq \| x - x_n \| + M(Su_n\,, Su) \\ &\leq \| x - x_n \| + \xi\;\, \| u_n - u \| \;\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $x \in Su$. Similarly $y \in Tu$, $z \in Au$. From (3.3), we have

$$g(u) = m(z) + J^{\partial\phi(.,\,u)}{}_{\eta}\,[\, g(u) - \eta\; N(x,y) - m(z)\,].$$

Therefore, it follows from Lemma 3.1 that the set of elements (u, x, y, z) is a solution of that problem (2.1). This completes the proof.

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