Mathematical and Computational Applications

MDPI

Article

Topology on Soft Continuous Function Spaces

Taha Yasin Öztürk [1,*,†] and Sadi Bayramov [2,*,†]

[1] Department of Mathematics; Faculty of Science and Letters, Kafkas University; TR-36100 Kars, Turkey
[2] Department of Algebra and Geometry, Baku State University, Baku Az 1148, Azerbaijan
* Correspondence: taha36100@kafkas.edu.tr (T.Y.O.), sbayram@bsu.edu.az (S.B.)
† These authors contributed equally to this work.

Academic Editor: Fazal M. Mahomed
Received: 28 December 2016; Accepted: 3 April 2017; Published: 27 April 2017

Abstract: The concept of soft sets was initiated by Molodtsov. Then, some operations on soft sets were defined by Maji et al. Later on, the concept of soft topological space was introduced. In this paper, we introduce the concept of the pointwise topology of soft topological spaces. Finally, we investigate the properties of soft mapping spaces and the relationships between some soft mapping spaces.

Keywords: soft set; soft point; soft topological space; soft continuous mapping; soft mapping spaces; soft pointwise topology

1. Introduction

Because classical methods can inhere troubles, most practical problems in different scientific fields such as engineering, social science, economics, environment, and medical science have required solutions via technical methods rather than dealing with classical methods. Insufficiency of the theories of parameterization tools may result in these difficulties. The concept of soft set theory was initiated by Molodtsov [1] as a new mathematical tool in order to deal with uncertainties. In addition, the works of Maji et al. [2,3] focus on operations over soft set. It can be said that the algebraic structure of set theories bearing on uncertainties is an important problem. Hence, many researchers have been interested in the algebraic structure of soft set theory, and there are many works on this subject. For example, soft groups and their basic properties were introduced by Aktaş and Çağman [4]. Later, initial concepts of soft rings were brought into attention by U. Acar et al. [5]. Then, establishing a connection between soft sets and semirings, F. Feng et al. [6] defined soft semirings and several related notions. Later on, M. Shabir et al. [7] worked on soft ideals over a semigroup. Qiu Mei Sun et al. [8] introduced soft modules and their basic properties. Continuing in this way from specific to more general, fuzzy soft modules and intuitionistic fuzzy soft modules were introduced by Gunduz and Bayramov [9,10], respectively, and they investigated some basic properties of these modules. Recently, chain complexes of soft modules and soft homology modules of them were defined by Ozturk and Bayramov [11], and then the concepts of inverse and direct systems in the category of soft modules were introduced by Ozturk et al. [12].

Recently, the study of soft topological spaces was initiated by Shabir and Naz [13]. The works [14–18] concentrated on the theoretical studies of soft topological spaces. The concepts of soft points defined in References [14–18] were different from those in [19]. In our work, we use the concept of soft point defined by Bayramov and Gunduz [19].

In the present study, the pointwise topology is defined in soft continuous mapping space, and the properties of soft mapping spaces are investigated. Subsequently, we give some relationships between some soft mappings spaces.

2. Preliminaries

Here we give necessary definitions and theorems for soft sets that have already been given in the literature. Thus, first of all, we present the definition of the soft set given by Molodtsov [1]. Throughout the study, we will assume X to be an initial universe set and E to be a set of parameters. Then, $P(X)$ will denote the power set of X.

Definition 1. *A pair (F,E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$* [1]. In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be considered as the set of $e-$elements of the soft set (F,E), or as the set of $e-$approximate elements of the soft set.

Definition 2. *For two soft sets (F,E) and (G,E) over X, (F,E) is called a soft subset of (G,E) if $\forall e \in E$, $F(e) \subseteq G(e)$* [2].
This relationship is denoted by $(F,E) \widetilde{\subseteq} (G,E)$. Similarly, (F,E) is called a soft superset of (G,E) if (G,E) is a soft subset of (F,E). This relationship is denoted by $(F,E) \widetilde{\supseteq} (G,E)$. Two soft sets (F,E) and (G,E) over X are called soft equal if (F,E) is a soft subset of (G,E), and (G,E) is a soft subset of (F,E).

Definition 3. *The intersection of two soft sets (F,E) and (G,E) over X is the soft set (H,E), where $\forall e \in E$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F,E) \widetilde{\cap} (G,E) = (H,E)$* [2].

Definition 4. *The union of two soft sets (F,E) and (G,E) over X is the soft set, where $\forall e \in E$, $H(e) = F(e) \cup G(e)$. This relationship is denoted by $(F,E) \widetilde{\cup} (G,E) = (H,E)$* [2].

Definition 5. *A soft set (F,E) over X is said to be a NULL soft set denoted by Φ if for all $e \in E$, $F(e) = \emptyset$(null set)* [2].

Definition 6. *A soft set (F,E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all $e \in E$, $F(e) = X$* [2].

Definition 7. *The difference (H,E) of two soft sets (F,E) and (G,E) over X , denoted by $(F,E) \setminus (G,E)$, is defined as $H(e) = F(e)/G(e)$ for all $e \in E$* [13].

Definition 8. *Let (F,E) be a soft set over X and Y be a non-empty subset of X. Then, the sub soft set of (F,E) over Y denoted by $({}^{Y}F,E)$, is defined as follows ${}^{Y}F(e) = Y \cap F(e)$, for all $e \in E$. In other words, $({}^{Y}F,E) = \widetilde{Y} \widetilde{\cap} (F,E)$* [13].

Definition 9. *Let (F,A) and (G,B) be two soft sets over X_1 and X_2, respectively, and $A, B \subseteq E$* [4]. *The cartesian product $(F,A) \times (G,B)$ is defined by $(F \times G)_{(A \times B)}$, where*

$$(F \times G)_{(A\times B)}(e,k) = F(e) \times G(k), \quad \forall (e,k) \in A \times B.$$

Definition 10. *Let τ be the collection of soft sets over X* [13]*; then, τ is said to be a soft topology on X if*

(1) $\Phi, \widetilde{X}$ belongs to τ;
(2) the union of any number of soft sets in τ belongs to τ;
(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X,τ,E) is called a soft topological space over X.

Definition 11. *Let (X,τ,E) be a soft topological space over X, then members of τ are said to be soft open sets in X* [13].

Proposition 1. *Let (X, τ, E) be a soft topological space over X. Then, the collection $\tau_e = \{F(e) : (F, E) \in \tau\}$ for each $e \in E$, defines a topology on X* [13].

Definition 12. *The complement of a soft set (F, E) is denoted by $(F, E)'$ and is defined by $(F, E)' = (F', E)$, where $F' : E \to P(X)$ is a mapping given by $F'(e) = X - F(e)$ for all $e \in E$* [13].

Definition 13. *Let (X, τ, E) be a soft topological space over X. A soft set (F, E) over X is said to be soft closed in X if its relative complement $(F, E)'$ belongs to τ* [13].

Definition 14. *Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X* [13]. *Then, the soft closure of (F, E) denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E). Clearly, $\overline{(F, E)}$ is the smallest soft closed set over X which contains (F, E).*

Definition 15. *Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point, denoted by (x_e, E), if for the element $e \in E$, $F(e) = \{x\}$, and $F(e') = \emptyset$ for all $e' \in E - \{e\}$ (briefly denoted by x_e)* [19].

Definition 16. *For two soft points x_e and $y_{e'}$ over a common universe X, we say that the points are different points if $x \neq y$ or $e \neq e'$. [19]*

Definition 17. *The soft point x_e is said to belong to the soft set (F, E) denoted by $x_e \ \widetilde{\in}\ (F, E)$ if $x_e(e) \in F(e)$; i.e., $x \subseteq F(e)$* [19].

Definition 18. *Let (X, τ, E) be a soft topological space over X* [19]. *A soft set $(F, E) \widetilde{\subseteq} (X, E)$ is called a soft neighborhood of the soft point $x_e \widetilde{\in} (F, E)$ if there exists a soft open set (G, E) such that $x_e \widetilde{\in} (G, E) \widetilde{\subseteq} (F, E)$.*

Definition 19. *Let (X, τ, E) and (Y, τ', E) be two soft topological spaces, and $f : (X, \tau, E) \to (Y, \tau', E)$ be a mapping* [20]. *For each soft neighbourhood (H, E) of $(f(x)_e, E)$, if there exists a soft neighbourhood (F, E) of x_e such that $f((F, E)) \widetilde{\subseteq} (H, E)$, then f is said to be soft continuous mapping at x_e.*

If f is soft continuous mapping for all x_e, then f is called soft continuous mapping.

Definition 20. *$\{(\varphi_i, \psi_i) : (X, \tau, E) \to (Y_i, \tau_i, E_i)\}_{i \in \Delta}$ is a family of soft mappings, and $\{(Y_i, \tau_i, E_i)\}_{i \in \Delta}$ is a family of soft topological spaces* [14]. *Then, the topology τ generated from the subbase $\delta = \left\{(\varphi_i, \psi_i)^{-1}_{i \in \Delta}(F_i, E_i) : (F_i, E_i) \in \tau_i, i \in \Delta\right\}$ is called the soft topology (or initial soft topology) induced by the family of soft mappings $\{(\varphi_i, \psi_i)\}_{i \in \Delta}$.*

Definition 21. *Let $\{(X_i, \tau_i, E_i)\}_{i \in \Delta}$ be a family of soft topological spaces* [14]. *Then, the initial soft topology on $X = \prod_{i \in \Delta} X_i$ generated by the family $\{(p_i, q_i)\}_{i \in \Delta}$ is called product soft topology on X. (here, (p_i, q_i) is the soft projection mapping from (X, E) to (X_i, E) $i \in \Delta$).*

The product soft topology is denoted by $\prod_{i \in \Delta} \tau_i$.

3. Topology on Soft Continuous Function Spaces

Let $\{(X_s, \tau_s, E)\}_{s \in S}$ be a family of soft topological spaces over the same parameters set E. We define a family of soft sets $\left(\prod\limits_{s \in S} X_s, E\right)$ as follows;

If $F_s : E \to P(X_s)$ is a soft set over X_s for each $s \in S$, then $\prod\limits_{s \in S} F_s : E \to P(\prod\limits_{s \in S} X_s)$ is defined by $\left(\prod\limits_{s \in S} F_s\right)(e) = \prod\limits_{s \in S} F_s(e)$. Let us consider the topological product $\left(\prod\limits_{s \in S} X_s, \prod\limits_{s \in S} \tau_s, \prod\limits_{s \in S} E_s\right)$ of a family of soft topological spaces $\{(X_s, \tau_s, E)\}_{s \in S}$. We take the restriction to the diagonal $\Delta \subset \prod\limits_{s \in S} E_s$ of each soft set $\prod_{s\in S} F : \prod_{s\in S} E_s \to P(\prod_{s\in S} X_s)$. Since there exists a bijection mapping between the diagonal Δ and the parameters set E, then the restrictions of soft sets are soft sets over E [21].

Let (X,τ,E) be a soft topological space, $\{(Y_s,\tau_s',E)\}_{s\in S}$ be a family of soft topological spaces, and $\{(f_s,1_E) : (X,\tau,E) \to (Y_s,\tau_s',E)\}_{s\in S}$ be a family of soft mappings. For each soft point $x_e \,\tilde{\in}\, (X,\tau,E)$, we define the soft mapping $f = \underset{s\in S}{\triangle} f_s : (X,\tau,E) \to \left(\prod_{s\in S} Y_s,\tau',E\right)$ by $f(x_e) = \{f_s(x_e)\}_{s\in S} = \{(f_s(x))_e\}_{s\in S}$. If $f : (X,\tau,E) \to \left(\prod_{s\in S} Y_s,\tau',E\right)$ is any soft mapping, then $f = \underset{s\in S}{\triangle} f_s$ is satisfied for the family of soft mappings $\{f_s = p_s \circ f : (X,\tau,E) \to (Y_s,\tau_s',E)\}_{s\in S}$ [21].

Theorem 2. $f : (X,\tau,E) \to \left(\prod_{s\in S} Y_s,\tau',E\right)$ *is soft continuous if and only if* $f_s = p_s \circ f : (X,\tau,E) \to (Y_s,\tau_s',E)$ *is soft continuous for each* $s \in S$.

Proof. $\Longrightarrow$ Let f be a soft continuous mapping. Since the soft mappings p_s are also continuous, the composite mapping will be continuous.

$\Longleftarrow$ Let $\left(F_{s_1} \times ... \times F_{s_n} \times \prod_{s\neq s_1...s_n} \widetilde{Y}_s, E\right)$ be an any soft base of product topology.

$$\begin{aligned} f^{-1}\left(F_{s_1} \times ... \times F_{s_n} \times \prod_{s\neq s_1...s_n} \widetilde{Y}_s, E\right) &= f^{-1}\left(p_{s_1}^{-1}(F_{s_1}) \cap ... \cap p_{s_n}^{-1}(F_{s_n}), E\right) \\ &= \left(f^{-1}p_{s_1}^{-1}(F_{s_1}) \cap ... \cap f^{-1}p_{s_n}^{-1}(F_{s_n}), E\right) \end{aligned}$$

Since the soft mappings $p_{s_1} \circ f, ..., p_{s_n} \circ f$ are soft continuous, the soft set

$$\left(f^{-1}p_{s_1}^{-1}(F_{s_1}) \cap ... \cap f^{-1}p_{s_n}^{-1}(F_{s_n}), E\right)$$

is soft open. Thus, $f : (X,\tau,E) \to \left(\prod_{s\in S} Y_s,\tau',E\right)$ is soft continuous. □

If $\left\{f_s : (X_s,\tau_s,E) \to (Y_s,\tau_s',E)\right\}_{s\in S}$ is a family of soft continuous mappings, then the soft mapping $\prod_{s\in S} f_s : \left(\prod_{s\in S} X_s,\tau,E\right) \to \left(\prod_{s\in S} Y_s,\tau',E\right)$ is soft continuous.

Now, let the family of soft topological spaces $\{(X_s,\tau_s,E)\}_{s\in S}$ be disjoint; i.e., $X_{s_1} \cap X_{s_2} = \varnothing$ for each $s_1 \neq s_2$. For the soft set $F : E \to \bigcup_{s\in S} X_s$ over the set E, define the soft set $F|_{X_s} : E \to X_s$ by

$$F|_{X_s}(e) = F(e) \cap X_s,\ \forall e \in E$$

and the soft topology τ define by

$$(F,E) \in \tau \Longleftrightarrow \left(F|_{X_s}, E\right) \in \tau_s.$$

It is clear that τ is a soft topology.

Definition 22. *A soft topological space* $\left(\bigcup_{s\in S} X_s,\tau,E\right)$ *is called the soft topological sum of the family of soft topological spaces* $\{(X_s,\tau_s,E)\}_{s\in S}$ *and denoted by* $\bigoplus_{s\in S}(X_s,\tau_s,E)$.

Let $(i_s,1_E) : (X_s,\tau_s,E) \to \bigoplus_{s\in S}(X_s,\tau_s,E)$ be an inclusion mapping for each $s \in S$. Since

$$\left(i_s,1_E\right)^{-1}(F,E) = \left(F|_{X_s}, E\right) \in \tau_s, \text{ for } \left(F,E\right) \in \tau,$$

$(i_s, 1_E)$ is soft continuous.

Let $\{(X_s, \tau_s, E)\}_{s\in S}$ be a family of soft topological spaces, (Y, τ', E) be a soft topological space, and $\left\{f_s : (X_s, \tau_s, E) \to (Y, \tau', E)\right\}_{s\in S}$ be a family of soft mappings. We define the soft function $f = \underset{s\in S}{\nabla} f_s : \underset{s\in S}{\oplus}(X_s, \tau_s, E) \to (Y, \tau', E)$ by $f(x_e) = f_s(x_e) = (f_s(x))_e$, where each soft point $x_e \,\widetilde{\in}\, \underset{s\in S}{\oplus}(X_s, \tau_s, E)$ can belong to a unique soft topological space (X_{s_0}, τ_{s_0}, E). If $f : \underset{s\in S}{\oplus}(X_s, \tau_s, E) \to (Y, \tau', E)$ is any soft mapping, then $\underset{s\in S}{\nabla} f_s = f$ is satisfied for the family of soft mappings $\left\{f_s = f \circ i_s : (X_s, \tau_s, E) \to (Y, \tau', E)\right\}_{s\in S}$.

Theorem 3. *The soft mapping* $f : \underset{s\in S}{\oplus}(X_s, \tau_s, E) \to (Y, \tau', E)$ *is soft continuous if and only if* $f_s = f \circ i_s : (X_s, \tau_s, E) \to (Y, \tau', E)$ *are soft continuous for each* $s \in S$.

Proof. $\Longrightarrow$ Let f be a soft continuous mapping. Since the soft mappings i_s are also continuous, the composite mapping will be continuous.

$\Longleftarrow$ Let $(F, E) \in \tau'$ be a soft open set. The soft set $f^{-1}(F, E)$ belongs to the soft topology $\underset{s\in S}{\oplus}\tau_s$ if and only if the soft set $\left(f^{-1}(F)\big|_{X_s}, E\right)$ belongs to τ_s. Since

$$\left(f^{-1}(F)\Big|_{X_s}, E\right) = i_s^{-1}\left(f^{-1}(F), E\right) = \left(i_s^{-1} \circ f^{-1}\right)(F, E) = f_s^{-1}(F, E) \in \tau_s,$$

f is soft continuous. □

Let $\left\{f_s : (X_s, \tau_s, E) \to (Y_s, \tau_s', E)\right\}_{s\in S}$ be a family of soft continuous mappings. We define the mapping $f = \underset{s\in S}{\oplus} f_s : \underset{s\in S}{\oplus}(X_s, \tau_s, E) \to \underset{s\in S}{\oplus}(Y_s, \tau_s', E)$ by $f(x_e) = f_s(x_e)$, where each soft point $x_e \,\widetilde{\in}\, \underset{s\in S}{\oplus}(X_s, \tau_s, E)$ belongs to (X_{s_0}, τ_{s_0}, E). It is clear that if each f_s is soft continuous, then f is also soft continuous.

Theorem 4. *Let* $\{(X_s, \tau_s, E)\}_{s\in S}$ *be a family of soft topological spaces. Then,*

$$\left(\prod_{s\in S} X_s, (\tau_s)_e\right) = \prod_{s\in S}\left(X_s, (\tau_s)_e\right) \text{ and } \left(\underset{s\in S}{\oplus} X_s, (\tau_s)_e\right) = \underset{s\in S}{\oplus}\left(X_s, (\tau_s)_e\right)$$

are satisfied for each $e \in E$.

Proof. We should show that $\tau_e = \prod_{s\in S}(\tau_s)_e$. Let us take any set U from τ_e. From the definition of the topology τ_e, there exists a soft open set

$$\left(\left(F_{s_1}, E\right) \times ... \times \left(F_{s_n}, E\right) \times \prod_{s\neq s_1...s_n} \widetilde{X}_s\right)$$

such that the set $U = \left(F_{s_1}(e) \times ... \times F_{s_n}(e) \times \prod_{s\neq s_1...s_n} X_s\right)$ belongs to the topology $\prod_{s\in S}(\tau_s)_e$.

Conversely, let $\left(U_{s_1} \times ... \times U_{s_n} \times \prod_{s\neq s_1...s_n} X_s\right) \in \prod_{s\in S}(\tau_s)_e$. Then, from the definition of the topology $(\tau_{s_i})_e$, there exist soft open sets $(F_{s_1}, E), ..., (F_{s_n}, E)$ such that $F_{s_1}(e) = U_{s_1}, ..., F_{s_n}(e) = U_{s_n}$. Then,

$$U_{s_1} \times ... \times U_{s_n} \times \prod_{s\neq s_1...s_n} X_s = F_{s_1}(e) \times ... \times F_{s_n}(e) \times \prod_{s\neq s_1...s_n} X_s \in \tau_e.$$

The topological sum can be proven in the same way. □

Let (X, τ, E) and $\left(Y, \tau^{'}, E\right)$ be two soft topological spaces. Y^X denotes the all soft continuous mappings from the soft topological space (X, τ, E) to the soft topological space $\left(Y, \tau^{'}, E\right)$; i.e.,

$$Y^X = \left\{(f, 1_E) : (X, \tau, E) \to \left(Y, \tau^{'}, E\right) \mid (f, 1_E) - \text{a soft continuous map}\right\}.$$

If (F, E) and (G, E) are two soft sets over X and Y, respectively, then we define the soft set $\left(G^F, E\right)$ over Y^X as follows:

$$G^F(e) = \left\{\left(f, 1_E\right) : \left(X, \tau, E\right) \to \left(Y, \tau^{'}, E\right) \mid f\left(F(e)\right) \subset G(e)\right\} \text{ for each } e \in E.$$

Now, let $x_\alpha \widetilde{\in} (X, \tau, E)$ be an any soft point. We define the soft mapping $e_{x_\alpha} : (Y^X, E) \to \left(Y, \tau^{'}, E\right)$ by $e_{x_\alpha}(f) = f(x_\alpha) = (f(x))_\alpha$. This mapping is called an evaluation map. For the soft set (G, E) over Y, $e_{x_\alpha}^{-1}(G, E) = (G^{x_\alpha}, E)$ is satisfied. The soft topology that is generated from the soft sets $\left\{(G^{x_\alpha}, E) \mid (G, E) \in \tau^{'}\right\}$ as a subbase is called a pointwise soft topology and denoted by τ_p.

Definition 23. $\left(Y^X, \tau_p, E\right)$ *is called a pointwise soft function space (briefly PISFS).*

Example 5. *Let $X = \{x^1, x^2\}$, $Y = \{y^1, y^2, y^3\}$ and $E = \{e_1, e_2\}$. If we give the soft sets $F_i : E \to P(X)$ for $i \in I$ and $G : E \to P(Y)$ defined by*

$$\begin{aligned}
F_1(e_1) &= \{x^1\},\ F_1(e_2) = \varnothing \\
F_2(e_1) &= \varnothing,\ F_2(e_2) = \{x^2\} \\
F_3(e_1) &= \{x^1\},\ F_3(e_2) = \{x^2\} \\
G(e_1) &= \{y^1, y^2\},\ G(e_2) = \{y^3\},
\end{aligned}$$

then the families $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ and $\tau^{'} = \{\Phi, \widetilde{Y}, (G, E)\}$ are soft topologies.

Now, let us give the soft continuous mappings set Y^X. $(f_i, 1_E) : (X, \tau, E) \to (Y, \tau^{'}, E)$ consist of the mappings

$$\begin{aligned}
f_1(x^1) &= y^1,\ f_1(x^2) = y^1 \\
f_2(x^1) &= y^2,\ f_2(x^2) = y^2 \\
f_3(x^1) &= y^3,\ f_3(x^2) = y^3 \\
f_4(x^1) &= y^1,\ f_4(x^2) = y^3 \\
f_5(x^1) &= y^2,\ f_5(x^2) = y^3.
\end{aligned}$$

Then, the soft subbase of soft pointwise topology consists of the following sets.

$$\begin{aligned}
(G^{x^1_{e_1}}, E) &= \{f_1, f_2, f_4, f_5\},\ (G^{x^1_{e_2}}, E) = \{f_3\} \\
(G^{x^2_{e_1}}, E) &= \{f_1, f_2\},\ (G^{x^2_{e_2}}, E) = \{f_3, f_4, f_5\}
\end{aligned}$$

Remark 5. *The evaluation mapping $e_{x_\alpha} : (Y^X, \tau_p, E) \to \left(Y, \tau^{'}, E\right)$ is a soft continuous mapping for each soft point $x_\alpha \widetilde{\in} (X, \tau, E)$.*

Proposition 6. *A soft mapping $g : (Z, \eta, E) \to (Y^X, \tau_p, E)$—where (Z, η, E) is a soft topological space—is a soft continuous mapping if and only if the soft mapping $e_{x_\alpha} \circ g : (Z, \eta, E) \to \left(Y, \tau^{'}, E\right)$ is a soft continuous mapping for each $x_\alpha \widetilde{\in} (X, \tau, E)$.*

Theorem 7. *If the soft topological space $\left(Y, \tau^{'}, E\right)$ is a soft T_i-space for each $i = 0, 1, 2$, then the soft space $\left(Y^X, \tau_p, E\right)$ is also a soft T_i-space.*

Proof. The soft points of the soft topological space $\left(Y^X, \tau_p, E\right)$ denoted by (f_α, E); i. e., if $\beta \neq \alpha$ then $f_\alpha(\beta) = \emptyset$ and if $\beta = \alpha$ then $f_\alpha(\beta) = f$. Now, let $f_\alpha \neq g_\beta$ be two soft points. Then, it should be $f \neq g$ or $\alpha \neq \beta$. If $f = g$, then $(f(x))_\alpha \neq (g(x))_\beta \mathrel{\widetilde{\in}} \left(Y, \tau^{'}, E\right)$ for each $x \in X$. If $f \neq g$, then $f(x^0) \neq g(x^0)$ such that $x^0 \in X$. Therefore, $\left(f(x^0)\right)_\alpha \neq \left(g(x^0)\right)_\beta$ is satisfied. In both cases, $\left(f(x^0)\right)_\alpha \neq \left(g(x^0)\right)_\beta \mathrel{\widetilde{\in}} \left(Y, \tau^{'}, E\right)$ is satisfied for at least one $x_0 \in X$. Since $\left(Y, \tau^{'}, E\right)$ is a soft T_i-space, there exists soft open sets $(F_1, E), (F_2, E) \in \tau^{'}$ where the condition of the soft T_i-space is satisfied. Then, the soft open sets $\left(F_1^{x^0_\alpha}, E\right) = e^{-1}_{x^0_\alpha}(F_1, E)$ and $\left(F_2^{x^0_\beta}, E\right) = e^{-1}_{x^0_\beta}(F_2, E)$ are neighbourhoods of soft points f_α and g_β, respectively, where the conditions of soft T_i-space are satisfied for these neighbourhoods. $\square$

Now, we construct relationships betwen some function spaces. Let $\{(X_s, \tau_s, E)\}_{s \in S}$ be a family of pairwise disjoint soft topological spaces, $\left(Y, \tau^{'}, E\right)$ be a soft topological space, and $\prod\limits_{s \in S}\left(Y^{X_s}, \tau_{s_p}, E\right)$, $\bigoplus\limits_{s \in S}(X_s, \tau_s, E)$ be a product and sum of soft topological spaces, respectively. The soft mapping

$$\nabla : \prod_{s \in S}\left(Y^{X_s}, \tau_{s_p}, E\right) \to \left(Y^{\bigoplus\limits_{s \in S} X_s}, \left(\tau^{'}\right)_p, E\right)$$

is defined by $\underset{s \in S}{\nabla}\left(\{f_s\}\right)(x_\alpha) = f_{s_0}(x_\alpha) = \left(f_{s_0}(x)\right)_\alpha$, where $\forall \{(f_s, 1_E)\} \in \prod\limits_{s \in S} Y^{X_s}$, $\forall x_\alpha \mathrel{\widetilde{\in}} \bigoplus\limits_{s \in S}(X_s, \tau_s, E)$, x_α belongs to unique (X_{s_0}, τ_{s_0}, E). We define the inverse mapping of ∇

$$\nabla^{-1} : \left(Y^{\bigoplus\limits_{s \in S} X_s}, \left(\tau^{'}\right)_p, E\right) \to \prod_{s \in S}\left(Y^{X_s}, \tau_{s_p}, E\right)$$

by $\nabla^{-1}(f) = \left\{f \circ i_s = f|_{X_s} : X_s \to Y\right\} \in \prod\limits_{s \in S}\left(Y^{X_s}, \tau_{s_p}, E\right)$ for each $f : \bigoplus\limits_{s \in S} X_s \to Y$.

Theorem 8. *The mapping*

$$\nabla : \prod_{s \in S}\left(Y^{X_s}, \tau_{s_p}, E\right) \to \left(Y^{\bigoplus\limits_{s \in S} X_s}, \left(\tau^{'}\right)_p, E\right)$$

is a soft homeomorphism in the pointwise soft topology.

Proof. To prove the theorem, it is sufficient to show that the mappings ∇ and ∇^{-1} are soft continuous. For this, we need to show that the soft set $\nabla^{-1}\left(e^{-1}_{x_\alpha}(F, E)\right)$ is a soft open set, where each $e^{-1}_{x_\alpha}(F, E)$ belongs to a soft subbase of the soft space $\left(Y^{\bigoplus\limits_{s \in S} X_s}, \left(\tau^{'}\right)_p, E\right)$.

$$e^{-1}_{x_\alpha}(F, E) = \left\{f : \bigoplus_{s \in S} X_s \to Y \mid f(x_\alpha) \mathrel{\widetilde{\in}} (F, E)\right\} = \left\{f_{s_0} : X_{s_0} \to Y \mid f_{s_0}(x_\alpha) \mathrel{\widetilde{\in}} (F, E)\right\}.$$

Since

$$\begin{aligned} \nabla^{-1}\left(e^{-1}_{x_\alpha}(F, E)\right) &= \nabla^{-1}\left(\left\{f_{s_0} : X_{s_0} \to Y \mid f_{s_0}(x_\alpha) \mathrel{\widetilde{\in}} (F, E)\right\}\right) \\ &= \left\{f_{s_0} : X_{s_0} \to Y \mid f_{s_0}(x_\alpha) \mathrel{\widetilde{\in}} (F, E)\right\} \times \prod_{s \neq s_0}\left(Y^{X_s}, \tau_{s_p}, E\right) \end{aligned}$$

is the last soft set, $\nabla^{-1}\left(e_{x_\alpha}^{-1}(F,E)\right)$ is a soft open set on the product space $\prod\limits_{s\in S}\left(Y^{X_s},\tau_{s_p},E\right)$.

Now, we prove that the mapping $\nabla^{-1}:\left(Y^{\underset{s\in S}{\oplus}X_s},\left(\tau'\right)_p,E\right)\to\prod\limits_{s\in S}\left(Y^{X_s},\tau_{s_p},E\right)$ is soft continuous. Indeed, for each, the soft set $\left(e_{x_\alpha}^{-1}(F,E)\right)_{s_0}(F,E)\times\prod\limits_{s\neq s_0}\left(Y^{X_s},\tau_{s_p},E\right)$ belongs to the subbase of the product space $\prod\limits_{s\in S}\left(Y^{X_s},\tau_{s_p},E\right)$,

$$\left(e_{x_\alpha}^{-1}\right)_{s_0}(F,E)\times\prod_{s\neq s_0}\left(Y^{X_s},\tau_{s_p},E\right)=\left\{\left\{f_s\right\}\in\prod_{s\in S}Y^{X_s}\mid f_{s_0}\left(x_\alpha\right)\tilde{\in}(F,E)\right\}$$

is satisfied.

Since the set

$$\begin{aligned}&\left(\nabla^{-1}\right)^{-1}\left(\left(e_{x_\alpha}^{-1}\right)_{s_0}(F,E)\times\prod_{s\neq s_0}\left(Y^{X_s},\tau_{s_p},E\right)\right)\\=\;&\nabla\left(\left(e_{x_\alpha}^{-1}\right)_{s_0}(F,E)\times\prod_{s\neq s_0}\left(Y^{X_s},\tau_{s_p},E\right)\right)\\=\;&\left\{\underset{s\in S}{\nabla}f_s:f_{s_0}\left(x_\alpha\right)\tilde{\in}(F,E)\right\}\end{aligned}$$

belongs to subbase of the soft topological space $\left(Y^{\underset{s\in S}{\oplus}X_s},\left(\tau'\right)_p,E\right)$, the mapping ∇^{-1} is soft continuous. Thus, the mapping

$$\nabla:\prod_{s\in S}\left(Y^{X_s},\tau_{s_p},E\right)\to\left(Y^{\underset{s\in S}{\oplus}X_s},\left(\tau'\right)_p,E\right)$$

is a soft homeomorphism. □

Now, let $\{(Y_s,\tau_s',E)\}_{s\in S}$ be the family of soft topological spaces, (X,τ,E) be a soft topological space. We define mapping

$$\Delta:\prod_{s\in S}\left(Y_s^X,\tau_{s_p}',E\right)\to\left(\left(\prod_{s\in S}Y_s\right)^X,\left(\prod_{s\in S}\tau_s'\right)_p,E\right)$$

by the rule $\forall\left\{f_s:X\to Y_s\right\}\in\prod\limits_{s\in S}\left(Y_s^X,\tau_{s_p}',E\right),\Delta\left\{f_s\right\}=\underset{s\in S}{\Delta}f_s$.

Let the inverse mapping $\Delta^{-1}=\left(\left(\prod\limits_{s\in S}Y_s\right)^X,\left(\prod\limits_{s\in S}\tau_s'\right)_p,E\right)\to\prod\limits_{s\in S}\left(Y_s^X,\tau_{s_p}',E\right)$ be

$$\Delta^{-1}(f)=\left\{p_s\circ f=f_s:X_s\to Y\right\}$$

for each $f\in\left(\left(\prod\limits_{s\in S}Y_s\right)^X,\left(\prod\limits_{s\in S}\tau_s'\right)_p,E\right)$.

Theorem 9. *The mapping*

$$\Delta:\prod_{s\in S}\left(Y_s^X,\tau_{s_p}',E\right)\to\left(\left(\prod_{s\in S}Y_s\right)^X,\left(\prod_{s\in S}\tau_s'\right)_p,E\right)$$

is a soft homeomorphism in the pointwise soft topology.

Proof. Since Δ is bijective mapping, to prove the theorem it is sufficient to show that the mappings Δ and Δ^{-1} are soft open. First, we show that the mapping Δ is soft open. Let us take an arbitrary soft set

$$\left(e^{-1}_{x_{\alpha_1}}\right)_{s_1}(F_{s_1},E)\times ...\times\left(e^{-1}_{x_{\alpha_k}}\right)_{s_k}(F_{s_k},E)\times\left(\prod_{s\neq s_1...s_k}Y_s\right)^X$$

belongs to the base of the product space $\prod_{s\in S}\left(Y_s^X,\tau'_{s_p},E\right)$. Since the soft set

$$\begin{aligned}&\Delta\left(\left(e^{-1}_{x_{\alpha_1}}\right)_{s_1}(F_{s_1},E)\times ...\times\left(e^{-1}_{x_{\alpha_k}}\right)_{s_k}(F_{s_k},E)\times\left(\prod_{s\neq s_1...s_k}Y_s\right)^X\right)\\ =\ &\left\{\left\{f_s\right\}\mid f_{s_1}\left(x^1_{\alpha_1}\right)\tilde{\in}\left(F_{s_1},E\right),...,f_{s_k}\left(x^k_{\alpha_k}\right)\tilde{\in}\left(F_{s_k},E\right)\right\}\\ =\ &(F_{s_1}^{x^1_{\alpha_1}},E)\times ...\times(F_{s_k}^{x^k_{\alpha_k}},E)\times\left(\prod_{s\neq s_1...s_k}Y_s\right)^X\end{aligned}$$

is soft open, Δ is a soft open mapping.

Similarly, it can be proven that Δ^{-1} is soft open mapping. Indeed, for each soft open set $e^{-1}_{x_\alpha}\left((F_{s_1},E)\times ...\times(F_{s_k},E)\times\prod_{s\neq s_1...s_k}Y_s\right)\in\left(\left(\prod_{s\in S}Y_s\right)^X,\left(\prod_{s\in S}\tau'_s\right)_p,E\right)$,

$$\begin{aligned}&\Delta^{-1}\left(e^{-1}_{x_\alpha}\left((F_{s_1},E)\times ...\times(F_{s_k},E)\times\prod_{s\neq s_1...s_k}Y_s\right)\right)\\ =\ &\Delta^{-1}\left(\left\{f:X\to\prod_{s\in S}Y_s\mid f(x_\alpha)\tilde{\in}(F_{s_1},E)\times ...\times(F_{s_k},E)\times\prod_{s\neq s_1...s_k}Y_s\right\}\right)\\ =\ &\left\{p_s\circ f\mid f(x_\alpha)\tilde{\in}(F_{s_1},E)\times ...\times(F_{s_k},E)\times\prod_{s\neq s_1...s_k}Y_s\right\}\\ =\ &\left\{p_s\circ f\mid p_{s_1}\circ f(x_\alpha)\tilde{\in}(F_{s_1},E),...,p_{s_k}\circ f(x_\alpha)\tilde{\in}(F_{s_k},E)\right\}.\end{aligned}$$

Hence this set is soft open and the theorem is proved. □

Now, let (X,τ,E), $\left(Y,\tau',E\right)$ and $\left(Z,\tau'',E\right)$ be soft topological spaces and $f:\left(Z,\tau'',E\right)\times(X,\tau,E)\to\left(Y,\tau',E\right)$ be a soft mapping. Then, the induced map $\hat{f}:X\to Y^Z$ is defined by $\hat{f}(x_\alpha)\left(z_\beta\right)=f\left(x_\alpha,z_\beta\right)$ for soft points $x_\alpha\tilde{\in}(X,\tau,E)$ and $z_\beta\tilde{\in}\left(Z,\tau'',E\right)$. We define exponential law

$$E:Y^{Z\times X}\to\left(Y^Z\right)^X$$

by using induced maps $E(f)=\hat{f}$; i.e., $E(f)(x_\alpha)\left(z_\beta\right)=f\left(z_\beta,x_\alpha\right)=\hat{f}(x_\alpha)\left(z_\beta\right)$. We define the following mapping

$$E^{-1}:\left(Y^Z\right)^X\to Y^{Z\times X}$$

which is an inverse mapping E as follows

$$E^{-1}(\hat{f})=f,\ \ E^{-1}(\hat{f})\left(z_\beta,x_\alpha\right)=\ E^{-1}\left(\hat{f}(x_\alpha)\left(z_\beta\right)\right)=f\left(z_\beta,x_\alpha\right).$$

Generally, in the pointwise topology for each soft continuous map g, the mapping $E^{-1}(g)$ need not be soft continuous. Let us give the solution of this problem under some conditions.

Theorem 10. *Let (X,τ,E), $\left(Y,\tau',E\right)$ and $\left(Z,\tau'',E\right)$ be soft topological spaces and the mapping $e : Y^Z \times X \to Z$, $e(f,z) = f(z)$ be soft continuous. If there is a pointwise soft topology in the function space Y^X and the soft mapping $\hat{g} : X \to Y^Z$ is soft continuous, then the soft mapping*

$$E^{-1}\left(\hat{g}\right) : Z \times X \to Y$$

is also soft continuous.

Proof. By using the mapping

$$1_Z \times \hat{g} : Z \times Y \to Z \times Y^Z,$$

we take

$$Z \times X \xrightarrow{1_Z \times \hat{g}} Z \times Y^Z \xrightarrow{t} Y^Z \times Z \xrightarrow{e} Y.$$

Hence $e \circ t \circ \left(1_Z \times \hat{g}\right) \in Y^{Z \times X}$, where t denotes switching mapping which is the mapping changing the places of the arguments. Let us apply exponential law E to $e \circ t \circ \left(1_Z \times \hat{g}\right)$. For each soft point $x_\alpha \tilde{\in} (X,\tau,E)$ and $z_\beta \tilde{\in} \left(Z,\tau'',E\right)$,

$$\begin{aligned}
\left\{\left[E\left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right)\right](x_\alpha)\right\}(z_\beta) &= \left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right)(z_\beta, x_\alpha) \\
&= e \circ t\left(z_\beta, \hat{g}(x_\alpha)\right) \\
&= e\left(\hat{g}(x_\alpha), z_\beta\right) \\
&= \left(\hat{g}(x_\alpha)\right)(z_\beta).
\end{aligned}$$

Since $E\left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right) = \hat{g}$, $E^{-1}\left(\hat{g}\right) = e \circ t \circ \left(1_Z \times \hat{g}\right)$. Hence evaluation maps e and t are soft continuous, $E^{-1}\left(\hat{g}\right)$ is soft continuous. □

4. Conclusions

In this paper, we introduce the concept of the pointwise topology of soft topological spaces. Finally, we investigate the properties of soft mapping spaces and the relationships between some soft mapping spaces. We hope that the results of this study may help in the investigation of soft normed spaces and in many studies.

Author Contributions: Taha Yasin Öztürk and Sadi Bayramov both equally contributed to this work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Molodtsov, D. Soft Set Theory-First Results. *Comput. Math. Appl.* **1999**, *37*, 19–31.
2. Maji, P.K.; Bismas, R.; Roy, A.R. Soft Set Theory. *Comput. Math. Appl.* **2003**, *45*, 555–562.
3. Maji, P.K.; Roy, A.R. An Application of Soft Sets in a Decision Making Problem. *Comput. Math. Appl.* **2002**, *44*, 1077–1083.
4. Aktas, H.; Çağman N. Soft Sets and Soft Group. *Inf. Sci.* **2007**, *177*, 2726–2735.
5. Acar, U.; Koyuncu, F.; Tanay, B. Soft Sets and Soft Rings. *Comput. Math. Appl.* **2010**, *59*, 3458–3463.

6. Feng, F.; Jun, Y.B.; Zhao, X. Soft Semirings. *Comput. Math. Appl.* **2008**, *56*, 2621–2628.
7. Shabir, M.; İrfan Ali, M. Soft ideals and generalized fuzzy ideals in semigroups. *New Math. Nat. Comput*. **2009**, *5*, 599–615.
8. Sun, Q.M.; Zhang, L.; Liu, J. Soft Sets and Soft Modules. *Lect. Notes Comput. Sci.* **2008**, *5009*, 403–409.
9. Gunduz, Ç.; Bayramov, S. Fuzzy Soft modules. *Int. Mat. Forum* **2011**, *6*, 517–527.
10. Gunduz, Ç.; Bayramov, S. Intuitionistic Fuzzy Soft Modules. *Comput. Math. Appl.* **2011**, *62*, 2480–2486.
11. Ozturk, T.Y.; Bayramov, S. Category of chain complexes of soft modules. *Int. Math. Forum* **2012**, *7*, 981–992.
12. Ozturk, T.Y.; Gunduz, C.; Bayramov, S. Inverse and direct systems of soft modules. *Ann. Fuzzy Math. Inf.* **2013**, *5*, 73–85.
13. Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799.
14. Aygunoğlu, A.; Agun, A. Some notes on soft topological spaces. *Neural Comput. Appl.* **2012**, *21*, 113–119.
15. Çağman, N.; Karataş, S.; Enginoğlu, S. Soft topology. *Comput. Math. Appl.* **2011**, 351–358.
16. Min, W.K. A note on soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 3524–3528.
17. Shabir, H.; Bashir, A. Some properties of soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 4058–4067.
18. Zorlutuna, İ.; Akdağ, M.; Min, W.K.; Atmaca, S. Remarks on soft topological spaces. *Ann. Fuzzy Math. Inf.* **2012**, *3*, 171–185.
19. Bayramov, S.; Gunduz, C. Soft locally compact spaces and soft paracompact spaces. *J. Math. Syst. Sci.* **2013**, *3*, 122–130.
20. Gunduz, A.C.; Sonmez, A.; Çakallı H. On soft Mappings. *arXiv* **2013**, arXiv:1305.4545.
21. Ozturk, T.Y.; Bayramov, S. Soft mappings space. *Sci. World J.* **2014**, *2014*, doi:10.1155/2014/307292.