Article

On Moments of Gamma—Exponentiated Functional Distribution

Katarzyna Górska 1, Andrzej Horzela 2 and Tibor K. Pogány 3,*

1 H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Division of Theoretical Physics, ul. Eliasza-Radzikowskiego 152, PL 31-342 Kraków, Poland; katarzyna.gorska@ifj.edu.pl
2 Faculty of Maritime Studies, University of Rijeka, Studentska 2, HR-51000 Rijeka, Croatia; andrzej.horzela@ifj.edu.pl
3 Institute of Applied Mathematics, Óbuda University, Bécsi út 96/b, H-1034 Budapest, Hungary
* Correspondence: poganj@pfri.hr

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Abstract: In this note we discuss the development of a new Gamma exponentiated functional GE(α, h) distribution, using the Gamma baseline distribution generating method by Zografos and Balakrishnan. The raw moments of the Gamma exponentiated functional GE(α, h) distribution are derived. The related probability distribution class is characterized in terms of Lambert W-function.

Keywords: Gamma-exponentiated functional distribution; moments; Lagrange-Bürmann inversion theorem; Lambert W–function; quantile function

MSC: 60E05, 62E15, 33C20

1. Introduction

The idea of adding a positive parameter in the exponent of the cumulative distribution function (cdf) for a continuous distribution was introduced by Lehmann [1], which results in a class of so-called exponentiated distribution, see also ([2], Chapter 2). Using exponentiated Gamma-type random variables (rv) (or Stacy’s generalized Gamma distribution [3]) for generating further distribution classes were used by Zografos and Balakrishnan ([4], p. 350 et seq.). Subsequently, considering a similar method Ristić and Balakrishnan ([5], p. 1192, Equation (2)) introduce a new family of distributions via the survival function of the general continuous baseline (or parent) distribution which turns out to be a “dual family of the Zografos–Balakrishnan family of distributions” ([5], p. 1192) with a set of three main motivations for their new distribution class [5], ibid. We point out that Ristić and Balakrishnan linked their study to the Gamma-exponentiated exponential distribution, which certain properties are discussed by Ristić and Nadarajah [6]; it is worth mentioning the companion note by Pogány [7] where their findings concerning moments are precised in terms of higher transcendental functions including confluent Fox–Wright generalized hypergeometric and generalized Hurwitz–Lerch zeta function.

Introducing two extra parameters Cordeiro et al. ([8], pp. 1–2) have covered both Lehmann I and Lehmann II type exponentiated distribution classes, calling these exponentiated generalized families, giving full consideration to certain special cases like exponentiated generalized Fréchet, Normal, Gamma and Gumbel distributions ([8], 2. Special Models). Numerous authors have linked to listed models considering special cases of Gamma generalized, exponentiated distribution classes, among others we refer to Gamma-exponentiated Weibull [9,10], exponentiated Weibull, exponentiated Pareto, exponentiated Gamma [11], Kumaraswamy generalized Gamma and Gumbel [12,13] distributions with exhaustive references lists and links to further sub–models and special cases, consult e.g., ([13], pp. 415–416); also see the recent article [14] where an extension is obtained for the generalized integro-exponential function.
by which the moment expression of the above listed distribution classes can be expressed in a closed or more compact form. Finally, we mention the related recent article [15] as well.

The main purpose of adding parameters to an existing distribution is to obtain classes of more flexible distributions which provide more adaptability in modeling various types of data. According to Zografos and Balakrishnan [4] the Gamma-exponentiated extended distribution possesses cdf \( F(x) \) given as

\[
F(x) = \frac{1}{\Gamma(a)} \int_{0}^{-\log G(x)} t^{a-1} e^{-t} \, dt, \quad a > 0, \quad x \in \mathbb{R},
\]

where the baseline distribution \( G \) has the survival function \( G(x) = 1 - G(x) \). The Gamma–exponentiated extended probability density function (pdf) related to Equation (1) can be expressed in the following form:

\[
f(x) = \frac{1}{\Gamma(a)} \left[-\log G(x)\right]^a - 1 \cdot G'(x), \quad a > 0, \quad x \in \mathbb{R}.
\]

The regularized Gamma function reads

\[
Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_{z}^{\infty} t^{a-1} e^{-t} \, dt, \quad a > 0,
\]

where \( \Gamma(a, x) \) denotes the upper incomplete Gamma function. Both, regularized Gamma and incomplete Gamma, are in-built in Mathematica under \texttt{GammaRegularized[a, z]} and \texttt{Gamma[a, z]}, respectively.

We specify the approach presented in [4] by choosing the baseline distribution’s survival function to be \( G(x) = \exp(-h(x)) \), where \( h: \mathbb{R}_+ \mapsto \mathbb{R}_+ \) denotes a nonnegative Borel function.

The rv \( X \) defined on a standard probability space \( (\Omega, \mathcal{F}, P) \), having cdf and pdf

\[
\begin{align*}
F(x) &= \left[1 - Q(a, \lambda x + \beta x^k)\right] 1_{\mathbb{R}_+}(x) \\
f(x) &= \frac{h'(x)}{\Gamma(a)} h^{a-1}(x) e^{-h(x)} 1_{\mathbb{R}_+}(x),
\end{align*}
\]

respectively, is called Gamma-exponentiated functional \( h \) distributed, signifying this \( X \sim \text{GE}(a, h) \). Here and in what follows, \( 1_A(x) \) denotes the indicator function of the set \( A \), i.e., \( 1_A(x) = 1 \) when \( x \in A \) and equals 0 elsewhere.

As an illustrative example of this approach can be the case considered by Pogány and Saboor [16] choosing \( h(x) = \lambda x + \beta x^k \) introduced the Gamma-exponentiated exponential Weibull distribution \( \text{GEEW}(\theta), \theta = (\lambda, \beta, k, a) > 0 \), which cdf and pdf are

\[
\begin{align*}
F(x) &= \left[1 - Q(a, \lambda x + \beta x^k)\right] 1_{\mathbb{R}_+}(x) \\
f(x) &= \frac{1}{\Gamma(a)} \left(\lambda + \beta k x^{k-1}\right) e^{-\lambda x - \beta x^k} \left(\lambda x + \beta x^k\right)^{a-1} 1_{\mathbb{R}_+}(x).
\end{align*}
\]

Finally, the incomplete Gamma function possesses a representation in terms of the Kummer’s confluent hypergeometric function ([17], Chapter 13)

\[
1F_1(a; b; z) = \sum_{n \geq 0} \frac{(a)_n z^n}{(b)_n n!},
\]

we have the equivalent form of the cdf

\[
F(x) = \frac{h^{a}(x)}{\Gamma(a + 1)} 1F_1(a; a + 1; -h(x)).
\]
Then for all we conclude where

**Theorem 1.** Let \( h \) be analytic monotone increasing function with \( h'(0) \neq 0 \) and let \( r.v. X \sim \text{GE}(a, h) \).

Then for all \( a > 0, r \geq 0 \) we have

\[
E X^r = \frac{r}{4 \pi^2} \oint_{\gamma_z} \oint_{\gamma_w} \frac{z^{r-1} [(1-w)^a - 1]}{h(z) w^2 (1-w)^a} \ \bar{F}_0 \left( 1, 1; -\frac{1}{h(z)w} \right) \ dz \ dw,
\]

where the positively oriented closed integration paths \( \gamma_z, \gamma_w \) are taken in a way that enclose the origins in the complex \( z \)-, and \( w \)-planes, respectively.

Moreover, under the same assumptions, we have

\[
E X^r = \frac{r}{2 \pi} \oint_{\gamma_z} \int_0^\infty \frac{z^{r-1} [(1-t)^{-\alpha} - 1]}{t} e^{-h(z) t} \ dz \ dt.
\]

**Proof.** Assume \( r > 0 \) and denoting \( h^{-1} \) the inverse of \( h \), we have

\[
E X^r = \int_0^\infty \frac{x^r}{\Gamma(a)} h'(x) h^{a-1}(x) e^{-h(x)} \ dx = \frac{1}{\Gamma(a)} \int_0^{\infty} \frac{(h^{-1}(t))' t^{a-1} e^{-t} \ dt}{t^a}.
\]

The Lagrange–Bürmann inversion theorem ([20], Equation (1.1) et seq.) reads:

Let \( a(z) = \sum_{n \geq 0} a_n z^n \), with \( a_1 \neq 0 \) (interpreted either as analytic function or a formal power series), and \( A(z) = \sum_{n \geq 0} A_n z^n \). Then

\[
A(a^{-1}(z)) = A_0 + \sum_{n \geq 1} \frac{z^n}{n} \left[ \frac{A'(\xi)}{a^{(n)}} \right] n;
\]

where \( [\xi^n] \) extracts the coefficient of \( \xi^n \) in a series: \( [\xi^n] (\sum_{k} c_k x^k) = c_m \).

Applying Equation (5) to the integrand of the moment \( E X^r \) above, being \( a \equiv h \) and \( A(\xi) = \xi^r, r \geq 0 \), we conclude

\[
E X^r = \frac{r}{\Gamma(a)} \sum_{n \geq 1} \frac{1}{n} \left( \int_0^\infty e^{x+n-1} e^{-t} \ dt \right) \left[ \frac{1}{h^{n}(\xi)} \right]^{r+n-1} \frac{1}{h^{n}(\xi)}
\]

\[
= \frac{r}{\pi} \sum_{n \geq 1} \frac{a_n}{n} \left[ \frac{1}{h^{n}(\xi)} \right]^{r+n-1} \frac{1}{h^{n}(\xi)}
\]

By the Cauchy differentiation formula we have

\[
E X^r = \frac{r}{\pi} \sum_{n \geq 0} \frac{(a + 1)n(1)n}{(2)n} \left[ \frac{1}{h^{n+1}(\xi)} \right]^{r+n} \frac{1}{h^{n+1}(\xi)}.
\]
\[ \left[ \zeta^n \right] \frac{\zeta^{r+n}}{h^{n+1}(\zeta)} = \frac{1}{n!} \frac{d^n}{d\zeta^n} \left( \frac{\zeta^{r+n}}{h^{n+1}(\zeta)} \right) \bigg|_{\zeta=0} = \frac{1}{2\pi i} \oint_{\gamma} \frac{z^{r-1}}{h^{n+1}(z)} \, dz, \]

that is
\[ EX' = \frac{r\alpha}{2\pi i} \oint_{\gamma} \frac{z^{r-1}}{h(z)} \sum_{n \geq 0} \frac{(a+1)_n(1)_n}{(2)_n} \frac{1}{h^n(z)} \, dz, \] (6)

where \( \gamma \) is a positively oriented simple integration path enclosing the origin. Having in mind the differentiation property of the Gauss’ hypergeometric function ([18], p. 28, (1.6.11))
\[ \frac{\partial^n}{\partial w^n} 2F_1(a, b; c; w) \bigg|_{w=0} = \frac{(a)_n(b)_n}{(c)_n} 2F_1(a+n, b+n; c+n; 0) = \frac{(a)_n(b)_n}{(c)_n}, \]
it follows also by the Cauchy’s differentiation formula:
\[ \frac{(a+1)_n(1)_n}{(2)_n} = \frac{\partial^n}{\partial w^n} 2F_1(a+1, 1; 2; w) \bigg|_{w=0} = \frac{n!}{2\pi i} \oint_{\gamma_w} 2F_1(a+1, 1; 2; w) \frac{1}{w^{n+1}} \, dw. \]

Choosing the integration paths \( \gamma_1, \gamma_0 \) according to the assumptions we get
\[ EX' = -\frac{r\alpha}{4\pi^2} \oint_{\gamma_0} \oint_{\gamma_w} \frac{z^{r-1}}{h(z)w} \sum_{n \geq 0} \frac{(1)_n}{(h(z)w)^n} \, dz \, dw, \]
which is in fact Equation (3) since
\[ 2F_1(a+1, 1; 2; w) = -\frac{(1-w)^a - 1}{aw(1-w)^a}. \] (7)

The rest is obvious.

As to Equation (4), we take the Laplace–integral formula ([18], p. 31, Equation (1.633)):
\[ 3F_1(\lambda, a, b; c; s^{-1}) = \frac{s^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-st} t^{\lambda-1} 2F_1(a, b; c; t) \, dt, \] (8)

which holds true for all \( a, b \in \mathbb{C}; \ c \in \mathbb{C} \setminus \mathbb{Z}^0 \) provided that \( \min \{ \Re(\lambda), \Re(s) \} > 0 \). Thus, starting from Equation (6), we transform the inner sum into a 3F_1 expression by Equation (8) and conclude
\[ EX' = \frac{r\alpha}{2\pi i} \oint_{\gamma} \frac{z^{r-1}}{h(z)} \sum_{n \geq 0} \frac{(a+1)_n(1)_n}{(2)_n} \frac{1}{h^n(z)} \, dz \]
\[ = \frac{r\alpha}{2\pi i} \oint_{\gamma} \frac{z^{r-1}}{h(z)} 3F_1 \left( 1, a+1, 1; 2; \frac{1}{h(z)} \right) \, dz \]
\[ = \frac{r\alpha}{2\pi i} \oint_{\gamma} \frac{z^{r-1}}{h(z)} \int_0^\infty e^{-h(z)t} 2F_1(a+1, 1; 2; t) \, dt \, dz. \] (9)

By Equation (7), the expression Equation (9) becomes
\[ EX' = \frac{r}{2\pi i} \oint_{\gamma} \int_0^\infty \frac{z^{r-1}}{t} \left[ (1-t)^{-a} - 1 \right] e^{-h(z)t} \, dt \, dz, \]
which completes the proof. \( \square \)

The consequence of Theorem 1 when \( r = 1 \), recalling that \( E X = a \), is
Corollary 1. For all $\alpha > 0$ we have
\[
\int_{\gamma_2} \int_{\gamma_w} (1 - w) a - 1 \frac{1}{z} \frac{\Gamma(2)}{2F_0(1, 1; -; \frac{1}{zw})} \; dw \; dz = 4\pi^2 \alpha.
\]
\[
\int_{\gamma_2} \int_{\gamma_w} \frac{[1 - t]^{-a} - 1}{t} \; e^{-zt} \; dz \; dt = 2\pi i \alpha.
\]
Here the integration contours $\gamma_2, \gamma_w$ remain the same as in Theorem 1.

The Lambert $W$-function is the inverse function of $W \mapsto W^W$. Its principal branch $W_p$ is the solution of $W^W = x$, for which $W_p(x) \geq W_p(-e^{-1})$. This function is in-built in Mathematica as ProductLog[z]. We are interested in $W_p$ exclusively for $x \geq 0$, where it is single–valued and monotone increasing, see [17], Section 4.13.

Any nondecreasing function $h$ possesses an generalized inverse
\[
h^-(x) := \inf\{t \in \mathbb{R}_+: h(t) \geq x\}, \quad t \in \mathbb{R}_+,
\]
with the convention that $\inf \emptyset = \infty$. Moreover, if $h$ is strong monotone increasing then $h^-$ coincides with the ‘ordinary’ inverse $h^{-1}$.

Theorem 2. Consider rv $Y = h^p(X) \exp(\sigma h(X)); \sigma, p \geq 0$, where $X \sim GE(\alpha, h)$. Then
\[
Y \sim GE(\alpha, h^-), \quad h^- := h^- \left[ \frac{P}{\sigma} W_p \left( \frac{\sigma}{p} x^{\frac{1}{p}} \right) \right].
\]
Moreover, for all $s \in (-\alpha p^{-1}, \sigma^{-1})$ we have
\[
EY^s = \frac{(a)^p s}{(1 - (\sigma s)^{1/p})},
\]
whenever $h: \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a nondecreasing Borel function.

Proof. The rv $X \sim GE(\alpha, h)$ possesses cdf $F_X$ in the form Equation (2). When $\sigma = 0$, then $Y \equiv h^p(X)$. Letting $\sigma > 0$, the pdf $F_Y$ of the rv $Y$ becomes
\[
F_Y(x) = P \left[ h(X) \exp \left( \frac{\sigma}{p} h(X) \right) < x^{\frac{1}{p}} \right] = P \left[ h(X) < \frac{P}{\sigma} W_p \left( \frac{\sigma}{p} x^{\frac{1}{p}} \right) \right]
\]
\[
= P \left\{ X < h^- \left[ \frac{P}{\sigma} W_p \left( \frac{\sigma}{p} x^{\frac{1}{p}} \right) \right] \right\} = F_X \left\{ h^- \left[ \frac{P}{\sigma} W_p \left( \frac{\sigma}{p} x^{\frac{1}{p}} \right) \right] \right\} \cdot 1_{\mathbb{R}_+}(x),
\]
which is equivalent to the first assertion Equation (10). In turn
\[
EY^s = Eh^p(X) \exp(\sigma s h(X)) = \frac{1}{F(\alpha)} \int_0^\infty h^{k+s-1}(x) e^{-(1-\sigma s)h(x)} \; dh(x),
\]
where the convergence of the integral is controlled by the condition $\sigma s < 1$ because $h$ is non–decreasing and positive at the infinity. Now, routine steps lead to the assertion. \hfill \square

The quantile function $Q_X$ of the rv $X \sim F(x)$ is defined as
\[
Q_X(p) = \inf\{x \in \mathbb{R}: p \leq F(x)\}, \quad p \in (0, 1).
\]
It is the generalized inverse of the cdf for a fixed probability $p$. The related result is the following
Theorem 3. Let \( X \sim GE(h) \), where \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) is a nonnegative monotone Borel function. Then the quantile function \( Q_X(p) \) reads

\[
Q_X(p) = h^{-1} \circ Q^{-1}(\alpha, 1 - p), \quad p \in (0, 1),
\]

where \( \circ \) denotes the composition of functions.

Proof. The quantile function is derived by inverting Equation (2). Therefore, for \( p \in (0, 1) \) fixed, solving the equation \( 1 - Q(\alpha, h(x)) = F(x) = p \) with respect to the regularized upper incomplete Gamma–function \( Q \), we get \( Q(\alpha, h(x)) = 1 - p \). Because \( \Gamma'(a, z) = -z^{a-1}e^{-z} < 0 \), the function \( \Gamma(a, z) = \Gamma(a) Q(a, z) \) is monotone in \( z \), therefore \( Q \) has an unique inverse \( Q^{-1} \):

\[
h(x) = Q^{-1}(\alpha, 1 - p).
\]

Remarking that \( h \) is monotone too, the proof is finished. \( \square \)

3. Concluding Remarks

In this manuscript, the authors discuss the development of a new distribution, Gamma exponentiated functional \( GE(\alpha, h) \) distribution, using the Gamma baseline distribution generating method by Zografos and Balakrishnan [4] and also related to the so called “dual family of the Zografos–Balakrishnan family of distributions” [5]. The main findings of the article are two equivalent complex path integral expressions for the raw moments of the Gamma exponentiated functional \( GE(\alpha, h) \) distribution derived in Theorem 1 by virtue of the generalized hypergeometric function \( 2F_0 \) in the integrand. By these results a master formula is derived for raw moments which are coming from the \( GE(\alpha, h) \) distribution family.

As an illustrative example for \( GE(\alpha, h) \) distribution serves the \( GEW(\theta) \) distributed rv considered recently by Pogány and Saboor [16]. We also refer to the exhaustive list of special cases listed in Introduction.

Finally, the related probability distribution class is characterized in terms of Lambert \( W \)–function in Theorem 2, while the quantile function is derived in Theorem 3 in terms of the regularized upper incomplete Gamma function \( Q \).

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References


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