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Testing for Equality of Parameters from Different Load-Sharing Systems

Stefan Bedbur * and Udo Kamps

Institute of Statistics, RWTH Aachen University, 52056 Aachen, Germany; udo.kamps@rwth-aachen.de
* Correspondence: bedbur@isw.rwth-aachen.de; Tel.: +49-241-80-94776

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Abstract: In reliability, sequential order statistics serve as a model for the component lifetimes of k-out-of-n systems, which are operating as long as k out of n components are operating. In contrast to modelling with order statistics, load-sharing effects and other impacts of failures on the performance of the remaining components may be taken into consideration. Inference for associated load-sharing parameters, as well as for the underlying baseline distribution, is then of particular interest. In a setup of multiple samples of sequential order statistics modelling the component lifetimes of possibly differently structured k-out-of-n systems, we provide exact statistical tests to check for common load-sharing or common baseline-distribution parameters. In the two-sample case, critical values for the corresponding test statistics are tabulated for small sample sizes, and the asymptotic distributions of the test statistics under the null hypotheses are derived. Based on a simulation study, power comparisons are addressed. The proposed tests may be applied to detect significant differences between systems or to decide whether a meta-analysis of the data may be conducted to increase the performance of subsequent inferential procedures.

Keywords: reliability; k-out-of-n system; load-sharing system; order statistics; sequential order statistics; likelihood-ratio test; Rao score test

MSC: 62F03; 62F05; 62N05

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1. Introduction

In reliability, the k-out-of-n system forms an important technical structure that comprises series and parallel systems as special cases. In such a system, n components work simultaneously and may fail over time. The system, however, is operating as long as at least k of the components are operating, such that the $(n-k+1)$-th component failure time coincides with the lifetime of the system. In a simple model, the component lifetimes are described, say, by n non-negative, independent, and identically distributed (iid) random variables with cumulative distribution function (cdf) F. The $(n-k+1)$-th order statistic based on F then represents the system's lifetime. To provide a more flexible modelling of the lifetimes of k-out-of-n systems and their components, sequential order statistics (SOSs) were introduced in References [1,2], which allow to describe the impacts of failed components on the residual lifetimes of the remaining components. In particular, so called load-sharing effects can be modelled arising in systems, where all components (equally) share the total load of the system, and every component failure is likely to increase the stress put on the surviving ones. Obviously, modelling with order statistics is not adequate in applications with relevant impacts.

When SOSs are used as a load-sharing model for a k-out-of-n system, statistical inference for unknown underlying quantities typically has to be performed, aiming at a good model fit. In a proportional hazard rate setup of SOSs, the uncertainty of the model is captured within some baseline cdf or within a finite number of positive model parameters, for which a variety of inferential results have been derived. Estimators of model parameters or distribution parameters of the baseline cdf of SOSs are provided in, e.g., References [3–10]. Statistical tests for single or vectors of the model or baseline-distribution parameters of SOSs are proposed in References [3,6,7,11–13], which, in particular, allow for model selection in the sense of whether order statistics have to be rejected for modelling in a given situation. More inferential results, along with the detailed model and structural properties, can be found in Reference [14]. For nonparametric estimation and testing with SOSs, we refer to References [15–17]. Bayesian inference for SOSs is discussed, for instance, in References [18–20].

In this paper, continuing existing work on inference with SOSs, we develop statistical tests for the comparison of multiple k-out-of-n systems in the sense of whether they have model parameters or baseline-distribution parameters in common. In reliability and the particular context of load-sharing systems, sample sizes are often small going along with, e.g., low estimator accuracy. The proposed tests may be applied, for instance, to decide for a meta-analysis, i.e., a joint analysis of the data to improve the performance of subsequent inferential procedures, such as maximum-likelihood estimation of the unknown parameters.

The remainder of this article is as follows. First, we introduce the SOS model and review some basic properties (Section 2). Based on multiple samples of SOSs modelling possibly differently structured systems, we then derive exact statistical tests to check for common load-sharing parameters (Section 3.1) and for common baseline-distribution parameters (Section 4.1). In the particular case of two samples, critical values for test statistics are provided for small sample sizes and, moreover, asymptotic results are addressed (Sections 3.2 and 4.2). A simulation study is carried out to compare the proposed tests in terms of power (Section 5), and a concluding section finally highlights the main findings (Section 6).

2. Model and Basic Properties

SOSs were introduced in References [1,2] by means of a triangular scheme of independent random variables and certain recursive formulas. Meanwhile, other possible definitions have been proposed in the literature, for instance, based on independent power-function-distributed random variables (see Reference [21]) or on counting processes (see Reference [15]; cf. References [22,23]). For our purposes here, a likelihood approach to the model is sufficient.

Let $F_1, \ldots, F_n$ be absolutely continuous cdfs with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$ and corresponding density functions $f_1, \ldots, f_n$. Ordered random variables $X_1, \ldots, X_n$ (defined on a probability space $(\Omega, \mathfrak{A}, P)$) are called SOSs based on $F_1, \ldots, F_n$ if their joint density function with respect to (wrt) the Lebesgue measure is given by

$$f^{X_1,\ldots,X_n}(x_1, \ldots, x_n) \;=\; n! \prod_{j=1}^{n} \left\{ \frac{1 - F_j(x_j)}{1 - F_j(x_{j-1})} \right\}^{n-j} \frac{f_j(x_j)}{1 - F_j(x_{j-1})}$$

for all real numbers $x_1 \leq \cdots \leq x_n$, where here and in the following, $x_0 \equiv -\infty$ for the sake of a simple representation. In that case, $X_1, \ldots, X_n$ form a Markov chain with transition probabilities

$$P(X_j > t \,|\, X_{j-1} = x_{j-1}) \;=\; \left\{ \frac{1 - F_j(t)}{1 - F_j(x_{j-1})} \right\}^{n-j+1}, \qquad t > x_{j-1}\,,$$

for $j \in \{2, \ldots, n\}$; see References [1,24]. In the distribution-theoretical sense, order statistics (based on F_1) are included in the model and result by setting $F_1 = \cdots = F_n$.

When choosing

$$F_j \;=\; 1 - (1 - F)^{\alpha_j}\,, \qquad 1 \leq j \leq n\,, \tag{1}$$

for some absolutely continuous cdf F, which is referred to as a baseline cdf, and positive numbers $\alpha_1, \ldots, \alpha_n$, we arrive at a semiparametric SOS model, in which $F_1, \ldots, F_n$ have proportional hazard rates, i.e.,

$$\lambda_{F_j} = \alpha_j \lambda_F, \qquad 1 \leq j \leq n.$$

The conditional hazard rate of X_j, given $X_{j-1} = x_{j-1}$, is then

$$\lambda_{X_j \,|\, X_{j-1}=x_{j-1}}(t) \;=\; (n-j+1)\,\alpha_j\,\lambda_F(t), \qquad t > x_{j-1},$$

for $j \in \{2, \ldots, n\}$. In a load-sharing context, model parameters $\alpha_1, \alpha_2, \ldots$ may describe increasing stress put on surviving components in case of component failures in the following sense. All components start working at hazard rate $\alpha_1 \lambda_F$. Then, upon the j-th component failure for $j = 1, 2, \ldots$, the hazard rate of any still-operating component changes from $\alpha_j \lambda_F$ to $\alpha_{j+1} \lambda_F$. An extensive account on the SOS model, including motivational aspects, structural results, and inference, is provided by Reference [14].

When the component lifetimes of a k-out-of-n system are recorded, the data are type-II right-censored if $k > 1$, such that marginal densities are naturally of some interest. Here, the marginal density of the first r ($\leq n$) SOSs based on $F_1, \ldots, F_n$ is given by

$$f^{X_1,\ldots,X_r}(x_1, \ldots, x_r) \;=\; \frac{n!}{(n-r)!} \prod_{j=1}^{r} \left\{ \frac{1 - F_j(x_j)}{1 - F_j(x_{j-1})} \right\}^{n-j} \frac{f_j(x_j)}{1 - F_j(x_{j-1})}, \qquad x_1 \leq \cdots \leq x_r,$$

which, by assuming Formula (1), simplifies to

$$f^{X_1,\ldots,X_r}(x_1, \ldots, x_r) \;=\; \frac{n!}{(n-r)!} \prod_{j=1}^{r} \left(\frac{1 - F(x_j)}{1 - F(x_{j-1})} \right)^{(n-j+1)\alpha_j} \frac{\alpha_j f(x_j)}{1 - F(x_j)}, \qquad x_1 \leq \cdots \leq x_r, \qquad (2)$$

where f denotes the density function of the baseline distribution; see, e.g., Reference [14]. For brevity, let $\mathcal{X}_F = \{(x_1, \ldots, x_r) : F^{-1}(0+) < x_1 \leq \cdots \leq x_r < F^{-1}(1)\}$ denote the sample space in what follows.

3. Testing for Equal Load-Sharing Parameters

For $1 \leq k \leq m$, we assume to have s_k observations of the first r component-failure times in an $(n_k - r + 1)$-out-of-n_k system. In sample $k \in \{1, \ldots, m\}$, failure times are described by iid vectors $\boldsymbol{X}_1^{(k)}, \ldots, \boldsymbol{X}_{s_k}^{(k)}$, where $\boldsymbol{X}_i^{(k)} = (X_{i1}^{(k)}, \ldots, X_{ir}^{(k)})$ is distributed as the first r SOSs based on a known absolutely continuous cdf $F^{(k)}$ with density function $f^{(k)}$ and unknown model parameters $\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}$ for $1 \leq i \leq s_k$ (see Formula (2) and Table 1).

Table 1. Illustration of the sampling scheme.

sample	1	...	m
system	$(n_1 - r + 1)$-out-of-n_1	...	$(n_m - r + 1)$-out-of-n_m
# observed systems	s_1	...	s_m
iid vectors of SOSs	$\boldsymbol{X}_1^{(1)}, \ldots, \boldsymbol{X}_{s_1}^{(1)}$ $\boldsymbol{X}_i^{(1)} = (X_{i1}^{(1)}, \ldots, X_{ir}^{(1)}), 1 \leq i \leq s_1$	(independent)	$\boldsymbol{X}_1^{(m)}, \ldots, \boldsymbol{X}_{s_m}^{(m)}$ $\boldsymbol{X}_i^{(m)} = (X_{i1}^{(m)}, \ldots, X_{ir}^{(m)}), 1 \leq i \leq s_m$
baseline cdf	$F^{(1)}$	...	$F^{(m)}$
model parameters	$\alpha_1^{(1)}, \ldots, \alpha_{n_1}^{(1)}$	...	$\alpha_1^{(m)}, \ldots, \alpha_{n_m}^{(m)}$

As an example, $F^{(k)}$ may be specified as the standard exponential distribution for all $k \in \{1, \ldots, m\}$ to assume exponential component lifetimes in all systems with varying scale parameters upon failures. Moreover, $\mathbf{X}_i^{(k)}$, $1 \leq i \leq s_k$, $1 \leq k \leq m$, are supposed to be independent, the joint density function of which can then be represented as

$$f_{\boldsymbol{\alpha}}(\mathbf{x}) \;=\; \exp\left\{\sum_{k=1}^{m}\sum_{j=1}^{r} \alpha_j^{(k)}\, T_j^{(k)}(\mathbf{x}) \;+\; \sum_{k=1}^{m} s_k \sum_{j=1}^{r} \log \alpha_j^{(k)}\right\} h(\mathbf{x})\,, \tag{3}$$

where, for $1 \leq k \leq m$ and $1 \leq j \leq r$,

$$T_j^{(k)}(\mathbf{x}) \;=\; (n_k - j + 1) \sum_{i=1}^{s_k} \log\left(\frac{1 - F^{(k)}(x_{ij}^{(k)})}{1 - F^{(k)}(x_{i,j-1}^{(k)})}\right),$$

and

$$h(\mathbf{x}) \;=\; \prod_{k=1}^{m}\left(\frac{n_k!}{(n_k - r)!}\right)^{s_k} \prod_{i=1}^{s_k}\prod_{j=1}^{r} \frac{f^{(k)}(x_{ij}^{(k)})}{1 - F^{(k)}(x_{ij}^{(k)})}$$

for $\mathbf{x} = (\boldsymbol{x}_1^{(1)}, \ldots, \boldsymbol{x}_{s_1}^{(1)}, \ldots, \boldsymbol{x}_1^{(m)}, \ldots, \boldsymbol{x}_{s_m}^{(m)})$ with $\boldsymbol{x}_i^{(k)} = (x_{i1}^{(k)}, \ldots, x_{ir}^{(k)}) \in \mathcal{X}_{F^{(k)}}$, $1 \leq i \leq s_k$, $1 \leq k \leq m$. Here, $\boldsymbol{\alpha}$ denotes the vector of all model parameters $\alpha_j^{(k)}$, $1 \leq j \leq r$, $1 \leq k \leq m$. It is well known that the statistics $T_j^{(k)}$, $1 \leq j \leq r$, $1 \leq k \leq m$, are independent with $-T_j^{(k)} \sim \Gamma(s_k, 1/\alpha_j^{(k)})$, i.e., a gamma distribution with shape parameter s_k and scale parameter $1/\alpha_j^{(k)}$; see, e.g., Reference [5]. Moreover, for $1 \leq k \leq m$ and $1 \leq j \leq r$, the maximum-likelihood estimator (MLE) of $\alpha_j^{(k)}$ is given by

$$\hat{\alpha}_j^{(k)} \;=\; -\,\frac{s_k}{T_j^{(k)}}\,, \tag{4}$$

which can directly be seen by computing the first two derivatives of the log-likelihood function corresponding to Formula (3); cf. References [5,7]. As a consequence, $\hat{\alpha}_j^{(k)}$, $1 \leq j \leq r$, $1 \leq k \leq m$, are independent and inverse-gamma-distributed. In what follows, the vector of these MLEs is denoted by $\hat{\boldsymbol{\alpha}}$.

3.1. Exact Tests

For $1 \leq j \leq r$, let $I_{j1}, \ldots, I_{jq_j}$ be nonempty index sets forming a partition of $\{1, \ldots, m\}$, and let $q_j < m$ for at least one index $j \in \{1, \ldots, r\}$. Then, Equation (3) can be rewritten as

$$f_{\boldsymbol{\alpha}}(\mathbf{x}) \;=\; \exp\left\{\sum_{j=1}^{r}\sum_{l=1}^{q_j}\sum_{k \in I_{jl}} \alpha_j^{(k)}\, T_j^{(k)}(\mathbf{x}) \;+\; \sum_{j=1}^{r}\sum_{l=1}^{q_j}\sum_{k \in I_{jl}} s_k \log \alpha_j^{(k)}\right\} h(\mathbf{x})\,.$$

We aim at testing the null hypothesis:

$$H_0: \quad \alpha_j^{(k)} = \alpha_j^{(\tilde{k})}\,, \quad k, \tilde{k} \in I_{jl}\,, \quad 1 \leq l \leq q_j\,, \quad 1 \leq j \leq r\,, \tag{5}$$

where, here and in the following, the alternative hypothesis is the negation of the null hypothesis (in the SOS model). Under null hypothesis (5), the j-th model parameter is assumed to be constant in blocks, where blocks I_{jl}, $1 \leq l \leq q_j$, refer to sample numbers, $1 \leq j \leq r$. Figure 1 illustrates such a situation. In the particular situation $q_1 = \cdots = q_r = 1$, the r model parameters are the same in all samples. For $m = 2$ and $q_1 = \cdots = q_r = 1$, null hypothesis (5) refers to the two-sample case with s_1 and s_2 observed systems, which then simply reads

$$H_0: \ \alpha_j^{(1)} = \alpha_j^{(2)}, \quad 1 \le j \le r.$$

Note that by setting $q_j = m$ for some $j \in \{1, \dots, r\}$, there is no assumption for model parameters $\alpha_j^{(1)}, \dots, \alpha_j^{(m)}$ under the null hypothesis.

		sample 1	2	3	4	5	6	q_j
j	1	$\alpha_1^{(1)}$	$\alpha_1^{(2)}$	$\alpha_1^{(3)}$	$\alpha_1^{(4)}$	$\alpha_1^{(5)}$	$\alpha_1^{(6)}$	1
	2	$\alpha_2^{(1)}$	$\alpha_2^{(2)}$	$\alpha_2^{(3)}$	$\alpha_2^{(4)}$	$\alpha_2^{(5)}$	$\alpha_2^{(6)}$	2
	3	$\alpha_3^{(1)}$	$\alpha_3^{(2)}$	$\alpha_3^{(3)}$	$\alpha_3^{(4)}$	$\alpha_3^{(5)}$	$\alpha_3^{(6)}$	2
	4	$\alpha_4^{(1)}$	$\alpha_4^{(2)}$	$\alpha_4^{(3)}$	$\alpha_4^{(4)}$	$\alpha_4^{(5)}$	$\alpha_4^{(6)}$	3
	5	$\alpha_5^{(1)}$	$\alpha_5^{(2)}$	$\alpha_5^{(3)}$	$\alpha_5^{(4)}$	$\alpha_5^{(5)}$	$\alpha_5^{(6)}$	6

Figure 1. Illustration of null hypothesis (5) for $m = 6$ and $r = 5$.

To derive the MLE of $\alpha_j^{(k)}$, $1 \le k \le m$, $1 \le j \le r$, under H_0, let τ_{jl} be any index in I_{jl} for $1 \le l \le q_j$ and $1 \le j \le r$. Then, we have to maximize expression

$$\exp\left\{ \sum_{j=1}^{r} \sum_{l=1}^{q_j} \alpha_j^{(\tau_{jl})} \sum_{k \in I_{jl}} T_j^{(k)}(\mathbf{x}) + \sum_{j=1}^{r} \sum_{l=1}^{q_j} \log \alpha_j^{(\tau_{jl})} \sum_{k \in I_{jl}} s_k \right\} h(\mathbf{x})$$

wrt $\alpha_j^{(\tau_{jl})}$, $1 \le l \le q_j$, $1 \le j \le r$. Analysis of the first two derivatives yields that the maximum is attained at

$$\tilde{\alpha}_j^{(\tau_{jl})} = -\frac{\sum_{k \in I_{jl}} s_k}{\sum_{k \in I_{jl}} T_j^{(k)}}$$

for $1 \le l \le q_j$ and $1 \le j \le r$. Hence, the MLE of $\alpha_j^{(k)}$, $1 \le k \le m$, $1 \le j \le r$, under H_0 is given by

$$\tilde{\alpha}_j^{(k)} = \tilde{\alpha}_j^{(\tau_{jl})}, \quad k \in I_{jl},$$

and the corresponding vector of estimators is denoted by $\tilde{\boldsymbol{\alpha}}$.

Now, the likelihood-ratio statistic $\Lambda = -2\log(f_{\tilde{\boldsymbol{\alpha}}}/f_{\hat{\boldsymbol{\alpha}}})$ for testing null hypothesis (5) is seen to be

$$\begin{aligned} \Lambda &= 2 \sum_{j=1}^{r} \sum_{l=1}^{q_j} \sum_{k \in I_{jl}} s_k \log\left(\frac{\hat{\alpha}_j^{(k)}}{\tilde{\alpha}_j^{(k)}} \right) \\ &= 2 \sum_{j=1}^{r} \sum_{l=1}^{q_j} \sum_{k \in I_{jl}} s_k \log\left(\frac{s_k}{\sum_{\tilde{k} \in I_{jl}} s_{\tilde{k}}} \, \frac{\sum_{\tilde{k} \in I_{jl}} T_j^{(\tilde{k})}}{T_j^{(k)}} \right). \end{aligned} \tag{6}$$

As a competing test statistic to Λ, we address the Rao score statistic $R = \boldsymbol{U}_{\tilde{\boldsymbol{\alpha}}}^t [\mathcal{F}(\tilde{\boldsymbol{\alpha}})]^{-1} \boldsymbol{U}_{\tilde{\boldsymbol{\alpha}}}$ for testing null hypothesis (5), which involves the score statistic $\boldsymbol{U}_{\boldsymbol{\alpha}} = \partial/\partial\boldsymbol{\alpha} \log f_{\boldsymbol{\alpha}}$ and the Fisher information matrix $\mathcal{F}(\boldsymbol{\alpha}) = E_{\boldsymbol{\alpha}}(\boldsymbol{U}_{\boldsymbol{\alpha}} \boldsymbol{U}_{\boldsymbol{\alpha}}^t)$, where superscript t denotes transposition, and integration is done wrt $f_{\boldsymbol{\alpha}}$; see, e.g., Reference [25]. In our model, $\mathcal{F}(\boldsymbol{\alpha})$ is a diagonal $(mr \times mr)$-matrix with blocks

$\mathbf{diag}(s_k/[\alpha_1^{(k)}]^2, \ldots, s_k/[\alpha_r^{(k)}]^2)$, $1 \le k \le m$, since $-T_j^{(k)} \sim \Gamma(s_k, 1/\alpha_j^{(k)})$, $1 \le j \le r$, $1 \le k \le m$, are independent. Hence, by using Formula (4),

$$\begin{aligned} R &= \sum_{j=1}^{r} \sum_{l=1}^{q_j} \sum_{k \in I_{jl}} \left(T_j^{(k)} + \frac{s_k}{\tilde{\alpha}_j^{(k)}} \right)^2 \frac{[\tilde{\alpha}_j^{(k)}]^2}{s_k} \\ &= \sum_{j=1}^{r} \sum_{l=1}^{q_j} \sum_{k \in I_{jl}} s_k \left(\frac{\tilde{\alpha}_j^{(k)}}{\hat{\alpha}_j^{(k)}} - 1 \right)^2 \\ &= \sum_{j=1}^{r} \sum_{l=1}^{q_j} \sum_{k \in I_{jl}} s_k \left(\frac{\sum_{\tilde{k} \in I_{jl}} s_{\tilde{k}}}{s_k} \frac{T_j^{(k)}}{\sum_{\tilde{k} \in I_{jl}} T_j^{(\tilde{k})}} - 1 \right)^2 . \end{aligned} \quad (7)$$

The likelihood-ratio test and the Rao score test then reject the null hypothesis if the corresponding test statistics exceed certain critical values, which have to be determined from the distributions of Λ and R under the null hypothesis and from the significance level of the tests. For this, the following theorem is useful.

Theorem 1. *For testing null hypothesis (5), Λ and R, given by Formulas (6) and (7), have single null distributions; i.e., under the null hypothesis, the distributions of both test statistics do not depend on the specific parameters.*

Proof. Λ and R depend on the data only through the ratios

$$V_j^{(k)} = \frac{\sum_{\tilde{k} \in I_{jl}} T_j^{(\tilde{k})}}{T_j^{(k)}}, \qquad k \in I_{jl}, \quad 1 \le l \le q_j, \qquad 1 \le j \le r,$$

with independent statistics $-T_j^{(k)} \sim \Gamma(s_k, 1/\alpha_j^{(k)})$, $1 \le j \le r$, $1 \le k \le m$. Evidently, $V_j^{(k)}$, $1 \le j \le r$, $1 \le k \le m$, are, in turn, independent. Moreover, under H_0, the statistics $T_j^{(\tilde{k})}$, $\tilde{k} \in I_{jl}$, have a common scale parameter for $1 \le l \le q_j$ and $1 \le j \le r$, which implies that the distribution of $V_j^{(k)}$ is free of $\boldsymbol{\alpha}$ for $1 \le j \le r$ and $1 \le k \le m$. This yields the assertion. □

As a consequence of Theorem 1, the exact critical values for Λ and R subject to a desired significance level can be obtained via Monte Carlo simulation by independently sampling from gamma distributions with scale parameters all equal to 1. For $m = 2$ systems, small sample sizes $s_1 \le s_2$, and number $p = |\{j \in \{1, \ldots, r\} : q_j = 1\}| \in \{1, \ldots, 4\}$ of pairs with matching model parameters under the null hypothesis, critical values are shown in Tables 2 and 3 for a significance level of 5%. Note that these values do not depend on n_k, $F^{(k)}$, $k = 1, 2$, or on r, and may, in particular, be used for a full statistical comparison of two arbitrary $(n_i - r + 1)$-out-of-n_i systems, $i = 1, 2$, with $r \in \{1, \ldots, 4\}$.

Table 2. Exact critical values for Λ when testing for null hypothesis (5) with $m = 2$, $p \in \{1, \ldots, 4\}$, $s_1 \in \{1, \ldots, 10\}$, $s_2 \in \{s_1, \ldots, 10\}$, and a significance level of 5% (simulation size per value: 2×10^7).

p	$s_1 \setminus s_2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	4.66	4.50	4.45	4.44	4.43	4.42	4.42	4.42	4.41	4.41
	2		4.30	4.23	4.20	4.18	4.17	4.17	4.16	4.16	4.16
	3			4.15	4.12	4.10	4.08	4.08	4.07	4.07	4.06
	4				4.08	4.05	4.04	4.03	4.02	4.02	4.02
	5					4.03	4.02	4.01	4.00	3.99	3.99
	6						4.00	3.99	3.98	3.98	3.97
	7							3.98	3.97	3.96	3.96
	8								3.96	3.96	3.95
	9									3.95	3.94
	10										3.94
2	1	7.22	6.99	6.92	6.90	6.88	6.88	6.87	6.87	6.86	6.86
	2		6.69	6.59	6.55	6.52	6.51	6.49	6.49	6.49	6.48
	3			6.47	6.42	6.38	6.37	6.36	6.35	6.34	6.34
	4				6.36	6.32	6.30	6.28	6.28	6.27	6.26
	5					6.28	6.26	6.24	6.23	6.23	6.22
	6						6.24	6.22	6.21	6.19	6.19
	7							6.20	6.19	6.18	6.17
	8								6.18	6.16	6.16
	9									6.16	6.15
	10										6.14
3	1	9.41	9.11	9.02	8.99	8.97	8.96	8.95	8.95	8.94	8.94
	2		8.72	8.59	8.53	8.50	8.48	8.47	8.46	8.45	8.45
	3			8.44	8.37	8.33	8.31	8.29	8.28	8.27	8.26
	4				8.29	8.24	8.22	8.20	8.18	8.18	8.17
	5					8.20	8.17	8.15	8.13	8.12	8.11
	6						8.13	8.11	8.10	8.08	8.08
	7							8.09	8.08	8.06	8.05
	8								8.05	8.04	8.04
	9									8.03	8.02
	10										8.01
4	1	11.43	11.07	10.96	10.92	10.89	10.88	10.87	10.86	10.86	10.86
	2		10.59	10.43	10.36	10.32	10.30	10.28	10.27	10.26	10.26
	3			10.25	10.16	10.11	10.08	10.06	10.05	10.04	10.03
	4				10.06	10.01	9.98	9.96	9.94	9.93	9.92
	5					9.95	9.91	9.89	9.87	9.86	9.85
	6						9.88	9.85	9.83	9.82	9.80
	7							9.82	9.81	9.79	9.78
	8								9.78	9.77	9.75
	9									9.75	9.74
	10										9.72

Table 3. Exact critical values for R when testing for null hypothesis (5) with $m = 2$, $p \in \{1, \ldots, 4\}$, $s_1 \in \{1, \ldots, 10\}$, $s_2 \in \{s_1, \ldots, 10\}$, and a significance level of 5% (simulation size per value: 2×10^7).

p	$s_1 \setminus s_2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	1.80	2.65	3.11	3.34	3.48	3.58	3.64	3.69	3.73	3.76
	2		2.63	2.77	2.86	3.01	3.13	3.22	3.28	3.33	3.38
	3			3.00	3.08	3.12	3.15	3.17	3.19	3.21	3.23
	4				3.19	3.25	3.28	3.30	3.32	3.33	3.34
	5					3.32	3.36	3.38	3.40	3.41	3.42
	6						3.40	3.43	3.45	3.46	3.47
	7							3.46	3.49	3.50	3.51
	8								3.51	3.53	3.54
	9									3.54	3.56
	10										3.58
2	1	2.87	4.17	5.13	5.71	6.09	6.36	6.56	6.70	6.82	6.92
	2		3.88	4.37	4.79	5.13	5.39	5.59	5.75	5.87	5.98
	3			4.47	4.68	4.88	5.06	5.21	5.34	5.45	5.55
	4				4.81	4.93	5.04	5.14	5.22	5.30	5.38
	5					5.02	5.11	5.18	5.24	5.30	5.35
	6						5.17	5.24	5.28	5.33	5.37
	7							5.28	5.33	5.37	5.41
	8								5.37	5.40	5.43
	9									5.43	5.47
	10										5.49
3	1	3.81	5.36	6.66	7.51	8.08	8.50	8.81	9.04	9.24	9.40
	2		5.14	5.73	6.32	6.79	7.16	7.45	7.67	7.87	8.02
	3			5.82	6.12	6.42	6.69	6.92	7.12	7.28	7.43
	4				6.24	6.41	6.59	6.76	6.90	7.04	7.16
	5					6.52	6.63	6.75	6.86	6.96	7.05
	6						6.71	6.80	6.88	6.96	7.03
	7							6.86	6.92	6.98	7.04
	8								6.97	7.02	7.06
	9									7.06	7.10
	10										7.13
4	1	4.75	6.46	8.01	9.07	9.80	10.35	10.75	11.08	11.35	11.55
	2		6.32	7.00	7.70	8.28	8.72	9.09	9.38	9.62	9.83
	3			7.11	7.47	7.84	8.17	8.46	8.71	8.92	9.10
	4				7.60	7.81	8.03	8.25	8.43	8.61	8.75
	5					7.92	8.06	8.21	8.36	8.49	8.60
	6						8.15	8.26	8.36	8.46	8.56
	7							8.32	8.40	8.48	8.56
	8								8.46	8.52	8.58
	9									8.56	8.61
	10										8.65

3.2. Asymptotic Tests

We address asymptotic results for $m = 2$ systems and null hypothesis

$$H_0: \quad \alpha_j^{(1)} = \alpha_j^{(2)}, \quad j \in J, \tag{8}$$

for some nonempty index set $J = \{j_1, \ldots, j_p\} \subseteq \{1, \ldots, r\}$, in the case of which Formulas (6) and (7) simplify to

$$\begin{aligned}
\Lambda &= 2\sum_{j\in J}\left[s_1 \log\left(\frac{s_1}{s_1+s_2}\,\frac{T_j^{(1)}+T_j^{(2)}}{T_j^{(1)}}\right) + s_2 \log\left(\frac{s_2}{s_1+s_2}\,\frac{T_j^{(1)}+T_j^{(2)}}{T_j^{(2)}}\right)\right] \\
&= 2\sum_{j\in J}\left[s_1 \log\left(\frac{s_1}{s_1+s_2}\,(1+1/Q_j)\right) + s_2 \log\left(\frac{s_2}{s_1+s_2}\,(1+Q_j)\right)\right] \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad R &= \sum_{j\in J}\left[\frac{1}{s_1}\left((s_1+s_2)\,\frac{T_j^{(1)}}{T_j^{(1)}+T_j^{(2)}} - s_1\right)^2 + \frac{1}{s_2}\left((s_1+s_2)\,\frac{T_j^{(2)}}{T_j^{(1)}+T_j^{(2)}} - s_2\right)^2\right] \\
&= \sum_{j\in J}\left[\frac{1}{s_1}\left(\frac{s_2T_j^{(1)}}{T_j^{(1)}+T_j^{(2)}} - \frac{s_1T_j^{(2)}}{T_j^{(1)}+T_j^{(2)}}\right)^2 + \frac{1}{s_2}\left(\frac{s_1T_j^{(2)}}{T_j^{(1)}+T_j^{(2)}} - \frac{s_2T_j^{(1)}}{T_j^{(1)}+T_j^{(2)}}\right)^2\right] \\
&= \sum_{j\in J}\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\left(\frac{s_2Q_j - s_1}{Q_j+1}\right)^2, \qquad (10)
\end{aligned}$$

where $Q_j = T_j^{(1)}/T_j^{(2)}$, $j \in J$, are independent statistics. Note that for $p = 1$ and $s_1 = s_2$, the likelihood-ratio test and the Rao score test are equivalent, since then

$$\begin{aligned}
(1+1/Q_{j_1})(1+Q_{j_1}) &= \frac{(T_{j_1}^{(1)}+T_{j_1}^{(2)})^2}{T_{j_1}^{(1)}T_{j_1}^{(2)}} \\
\text{and}\quad \left(\frac{Q_{j_1}-1}{Q_{j_1}+1}\right)^2 &= 1 - 4\left(\frac{(T_{j_1}^{(1)}+T_{j_1}^{(2)})^2}{T_{j_1}^{(1)}T_{j_1}^{(2)}}\right)^{-1},
\end{aligned}$$

and, hence, Λ and R are both strictly monotone functions of statistic $(T_{j_1}^{(1)}+T_{j_1}^{(2)})^2/(T_{j_1}^{(1)}T_{j_1}^{(2)})$.

As an overall assumption in this section, let $s_1/(s_1+s_2) \to a \in (0,1)$ when the total sample size increases. Then, by the strong law of large numbers,

$$Q_j \to \frac{a}{1-a}\,\frac{\alpha_j^{(2)}}{\alpha_j^{(1)}} \quad \text{almost surely (a.s.)}\,, \quad j \in J\,. \qquad (11)$$

Moreover, the central limit theorem yields that

$$\left(\frac{T_{j_1}^{(1)}+s_1/\alpha_{j_1}^{(1)}}{\sqrt{s_1}/\alpha_{j_1}^{(1)}}, \ldots, \frac{T_{j_p}^{(1)}+s_1/\alpha_{j_p}^{(1)}}{\sqrt{s_1}/\alpha_{j_p}^{(1)}}, \frac{T_{j_1}^{(2)}+s_2/\alpha_{j_1}^{(2)}}{\sqrt{s_2}/\alpha_{j_1}^{(2)}}, \ldots, \frac{T_{j_p}^{(2)}+s_2/\alpha_{j_p}^{(2)}}{\sqrt{s_2}/\alpha_{j_p}^{(2)}}\right) \xrightarrow{d} \mathcal{N}_{2p}(\mathbf{0}, \mathbf{I}_{2p})\,, \qquad (12)$$

where $\mathbf{I}_k$ is the unity (or identity) matrix in $\mathbb{R}^{k\times k}$, $\mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the k-dimensional normal distribution with mean $\boldsymbol{\mu} \in \mathbb{R}^k$ and positive definite covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{k\times k}$, and $\xrightarrow{d}$ means convergence in distribution.

Lemma 1. *Under null hypothesis (8),*

$$\frac{1}{\sqrt{s_1}}\left(s_2Q_{j_1} - s_1, \ldots, s_2Q_{j_p} - s_1\right) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p/(1-a))\,.$$

Proof. Since $Q_{j_1}, \ldots, Q_{j_p}$ are independent, it is sufficient to show that, under H_0, $(s_2 Q_j - s_1)/\sqrt{s_1} \xrightarrow{d} \mathcal{N}_1(0, 1/(1-a))$ for $j \in J$. Let $j \in J$ and $\alpha_j^{(1)} = \alpha_j^{(2)} = \alpha_j$, say. Then, we have

$$\begin{aligned}\frac{s_2 Q_j - s_1}{\sqrt{s_1}} &= \frac{T_j^{(1)}/\sqrt{s_1} - \sqrt{s_1}T_j^{(2)}/s_2}{T_j^{(2)}/s_2} \\ &= \frac{(T_j^{(1)} + s_1/\alpha_j)/(\sqrt{s_1}/\alpha_j) - \sqrt{s_1/s_2}(T_j^{(2)} + s_2/\alpha_j)/(\sqrt{s_2}/\alpha_j)}{\alpha_j T_j^{(2)}/s_2} .\end{aligned}$$

Since $s_1/s_2 \to a/(1-a)$ and $\alpha_j T_j^{(2)}/s_2 \to 1$ a.s. by the strong law of large numbers, Formula (12) along with the multivariate Slutsky theorem (see, e.g., Reference [25], Theorem 3.4.3) then yield the assertion. □

Theorem 2. *Under null hypothesis (8), Λ and R, given by Formulas (9) and (10), are asymptotically $\chi^2(p)$-distributed, i.e., chi-square-distributed with p degrees of freedom.*

Proof. Since $Q_{j_1}, \ldots, Q_{j_p}$ are independent, it is sufficient to show that, under H_0, any term of the sum in Formulas (9) and (10), respectively, is asymptotically chi-square-distributed with one degree of freedom. The assertion then follows by application of the continuous mapping theorem (see, e.g., Reference [26], Theorem 1.10). To this end, let H_0 be true and $j \in J$. Moreover, let $A = s_2 Q_j - s_1$ to simplify notation. From Taylor's theorem, we have for every $x > 0$ the identity

$$2\log(x) = (x-1)[2-(x-1)] + \frac{2}{3}\left(\frac{x-1}{\zeta}\right)^3 ,$$

where ζ lies in the interval with boundary points 1 and x. Application to both logarithmic arguments in Formula (9) yields

$$\begin{aligned}&2s_1 \log\left(\frac{s_1}{s_1+s_2}(1+1/Q_j)\right) + 2s_2 \log\left(\frac{s_2}{s_1+s_2}(1+Q_j)\right) \\ =\ & \frac{s_2 A}{s_1+s_2}\left(2 - \frac{A}{s_1+s_2}\right) - \frac{s_1 A/Q_j}{s_1+s_2}\left(2 + \frac{A/Q_j}{s_1+s_2}\right) + B \\ =\ & \frac{A/Q_j}{s_1+s_2}\left(2s_2 Q_j - \frac{s_2 A Q_j}{s_1+s_2} - 2s_1 - \frac{s_1 A/Q_j}{s_1+s_2}\right) + B \\ =\ & \frac{A^2/Q_j}{s_1+s_2}\left(2 - \frac{s_2 Q_j}{s_1+s_2} - \frac{s_1/Q_j}{s_1+s_2}\right) + B \qquad (13)\end{aligned}$$

with

$$B = \frac{2}{3}\left(\frac{A}{s_1+s_2}\right)^3 \left(\frac{1}{\zeta_2^3} - \frac{1}{\zeta_1^3 Q_j^3}\right) ,$$

where ζ_1 and ζ_2 lie in the interval with boundary points 1 and $s_1(1+1/Q_j)/(s_1+s_2)$, respectively $s_2(1+Q_j)/(s_1+s_2)$. By Formula (11) with $\alpha_j^{(1)} = \alpha_j^{(2)}$, the term in brackets in Formula (13) as well as ζ_1 and ζ_2 converge to 1 a.s.. Since

$$\frac{A}{s_1+s_2} = \left(\frac{A}{\sqrt{s_1}}\right) \frac{1}{\sqrt{s_1}} \frac{s_1}{s_1+s_2} ,$$

Lemma 1 and Formula (11), together with Slutsky's theorem, then yield that B converges to 0 in distribution, which implies this convergence in probability, and, moreover,

$$\frac{A^2/Q_j}{s_1+s_2} = \left(\frac{A}{\sqrt{s_1}}\right)^2 \frac{s_1/Q_j}{s_1+s_2} \xrightarrow{d} \chi^2(1)\,.$$

Application of Slutsky's theorem to Formula (13), twice, then yields the assertion for the likelihood-ratio statistic.

To show the assertion for the Rao score statistic, we rewrite the terms in Formula (10) as

$$\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\left(\frac{A}{Q_j+1}\right)^2 = \left(\frac{A}{\sqrt{s_1}}\right)^2 \left[\frac{s_1+s_2}{s_2}\left(\frac{1}{Q_j+1}\right)^2\right].$$

By Formula (11), the term in square brackets converges to $1-a$ a.s.. Application of Lemma 1 and Slutsky's theorem, again, then completes the proof. □

4. Testing for Equal Baseline-Distribution Parameters

We assume to have the sample situation as introduced at the beginning of Section 3 with the difference that model parameters $\alpha_j^{(k)}$, $1 \le j \le r$, and $1 \le k \le m$ are known, and parameters of the baseline cdfs $F^{(1)}, \dots, F^{(m)}$ are unknown. More precisely, for $k \in \{1,\dots,m\}$, let baseline cdf in sample k be of the form

$$F^{(k)}(x) = 1-\exp\{-\sigma_k\, g_k(x)\}\,, \quad x \ge 0\,, \tag{14}$$

for some unknown positive scale parameter σ_k and a known increasing function $g_k : [0,\infty) \to [0,\infty)$ with $g_k(0) = 0$ and $\lim_{x\to\infty} g_k(x) = \infty$, which is differentiable on $(0,\infty)$; cf. References [7,14,27]. As two examples, choices $g_k(x) = x$ and $g_k(x) = \log(x+1)$ for $x \ge 0$ correspond to an exponential and a Pareto baseline distribution, respectively. Hence, the uncertainty of the model is totally captured within the vector $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_m)$ of baseline-distribution parameters. Here, the joint density function of $\mathbf{X}_i^{(k)}$, $1 \le i \le s_k$, $1 \le k \le m$, can be written as

$$f_{\boldsymbol{\sigma}}(\mathbf{x}) = \exp\left\{\sum_{k=1}^m \sigma_k\, \tilde{T}^{(k)}(\mathbf{x}) + r\sum_{k=1}^m s_k \log\sigma_k\right\} \tilde{h}(\mathbf{x})\,,$$

where, for $1 \le k \le m$,

$$\tilde{T}^{(k)}(\mathbf{x}) = -\sum_{j=1}^r (n_k-j+1)\,\alpha_j^{(k)} \sum_{i=1}^{s_k} (g_k(x_{ij}^{(k)}) - g_k(x_{i,j-1}^{(k)}))\,,$$

and

$$\tilde{h}(\mathbf{x}) = \prod_{k=1}^m \left(\frac{n_k!}{(n_k-r)!}\right)^{s_k} \prod_{j=1}^r \left(\alpha_j^{(k)}\right)^{s_k} \prod_{i=1}^{s_k} g_k'(x_{ij}^{(k)})$$

for $\mathbf{x} = (\boldsymbol{x}_1^{(1)},\dots,\boldsymbol{x}_{s_1}^{(1)},\dots,\boldsymbol{x}_1^{(m)},\dots,\boldsymbol{x}_{s_m}^{(m)})$ with $\boldsymbol{x}_i^{(k)} = (x_{i1}^{(k)},\dots,x_{ir}^{(k)}) \in \mathcal{X}_{F^{(k)}}$, $1 \le i \le s_k$, $1 \le k \le m$ (cf. Formula (3)). Statistics $\tilde{T}^{(k)}$, $1 \le k \le m$, are independent, with $-\tilde{T}^{(k)} \sim \Gamma(rs_k, 1/\sigma_k)$ for $1 \le k \le m$. Moreover, for $1 \le k \le m$, the MLE of σ_k is given by

$$\hat{\sigma}_k = -\frac{rs_k}{\tilde{T}^{(k)}}\,,$$

as analysis of the first two derivatives of the corresponding log-likelihood function shows. In the following, let $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1,\dots,\hat{\sigma}_m)$.

4.1. Exact Tests

Let $I_1, \ldots, I_q$ with $q < m$ be nonempty index sets forming a partition of $\{1, \ldots, m\}$. Equation (4) can be rewritten as

$$f_{\sigma}(\mathbf{x}) \;=\; \exp\left\{\sum_{l=1}^{q}\sum_{k\in I_l}\sigma_k\,\tilde{T}^{(k)}(\mathbf{x}) \;+\; r\sum_{l=1}^{q}\sum_{k\in I_l}s_k\log\sigma_k\right\}\,\tilde{h}(\mathbf{x})\,.$$

We consider the test problem with null hypothesis

$$H_0: \quad \sigma_k = \sigma_{\tilde{k}}\,, \quad k,\tilde{k}\in I_l\,, \quad 1\le l\le q\,, \tag{15}$$

and develop the corresponding likelihood-ratio test and Rao score test as in Section 3.1. To derive the MLE of σ_k, $1 \le k \le m$, under H_0, let τ_l be any index in I_l for $1 \le l \le q$. Then, the aim is to maximize the term

$$\exp\left\{\sum_{l=1}^{q}\sigma_{\tau_l}\sum_{k\in I_l}\tilde{T}^{(k)}(\mathbf{x}) \;+\; r\sum_{l=1}^{q}\log\sigma_{\tau_l}\sum_{k\in I_l}s_k\right\}\,\tilde{h}(\mathbf{x})$$

wrt $\sigma_{\tau_1}, \ldots, \sigma_{\tau_q}$, the maximum of which is attained at

$$\tilde{\sigma}_{\tau_l} \;=\; -\,\frac{r\sum_{k\in I_l}s_k}{\sum_{k\in I_l}\tilde{T}^{(k)}}\,, \qquad 1\le l\le q\,.$$

Hence, the MLE of σ_k, $1 \le k \le m$, under H_0 is given by

$$\tilde{\sigma}_k \;=\; \tilde{\sigma}_{\tau_l}\,, \quad k\in I_l\,.$$

Proceeding along the lines in Section 3.1, the likelihood-ratio statistic and Rao score statistic for testing null hypothesis (15) turn out to be

$$\tilde{\Lambda} \;=\; 2r\sum_{l=1}^{q}\sum_{k\in I_l}s_k\log\left(\frac{s_k}{\sum_{\tilde{k}\in I_l}s_{\tilde{k}}}\,\frac{\sum_{\tilde{k}\in I_l}\tilde{T}^{(\tilde{k})}}{\tilde{T}^{(k)}}\right) \tag{16}$$

$$\tilde{R} \;=\; r\sum_{l=1}^{q}\sum_{k\in I_l}s_k\left(\frac{\sum_{\tilde{k}\in I_l}s_{\tilde{k}}}{s_k}\,\frac{\tilde{T}^{(k)}}{\sum_{\tilde{k}\in I_l}\tilde{T}^{(\tilde{k})}}-1\right)^2, \tag{17}$$

Theorem 3. *For testing null hypothesis (15), $\tilde{\Lambda}$ and $\tilde{R}$, given by Formulas (16) and (17), have single null distributions (cf. Theorem 1).*

Proof. The assertion can be shown by using similar arguments as in the proof of Theorem 1. □

Theorem 3 allows for computing exact critical values for $\tilde{\Lambda}$ and $\tilde{R}$, subject to a desired significance level, by using Monte Carlo simulations and independently sampling from gamma distributions with scale parameters all equal to 1. For $m = 2$ systems, small sample sizes $s_1 \le s_2$, and $r \in \{1, \ldots, 4\}$, Tables 4 and 5 show such critical values for a significance level of 5%. Note that these values depend neither on n_k or g_k, $k = 1, 2$, nor on prespecified model parameters $\alpha_j^{(k)}$, $1 \le j \le r$, $1 \le k \le m$.

Table 4. Exact critical values for $\tilde{\Lambda}$ when testing for null hypothesis (15) with $m = 2$, $r \in \{1, \ldots, 4\}$, $s_1 \in \{1, \ldots, 10\}$, $s_2 \in \{s_1, \ldots, 10\}$, and a significance level of 5% (simulation size per value: 2×10^7).

r	$s_1 \setminus s_2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	4.66	4.50	4.45	4.44	4.43	4.42	4.42	4.42	4.42	4.41
	2		4.30	4.23	4.20	4.18	4.17	4.17	4.17	4.16	4.16
	3			4.16	4.11	4.10	4.09	4.08	4.07	4.07	4.07
	4				4.08	4.06	4.04	4.03	4.03	4.02	4.02
	5					4.03	4.02	4.00	4.00	3.99	3.99
	6						4.00	3.99	3.98	3.97	3.97
	7							3.98	3.97	3.96	3.96
	8								3.96	3.95	3.95
	9									3.95	3.94
	10										3.94
2	1	4.30	4.20	4.18	4.16	4.16	4.16	4.16	4.16	4.15	4.15
	2		4.08	4.04	4.03	4.02	4.01	4.01	4.01	4.01	4.01
	3			4.00	3.98	3.97	3.96	3.96	3.96	3.96	3.95
	4				3.96	3.95	3.95	3.94	3.94	3.93	3.93
	5					3.94	3.93	3.93	3.92	3.92	3.92
	6						3.92	3.91	3.91	3.91	3.91
	7							3.91	3.91	3.90	3.90
	8								3.90	3.90	3.89
	9									3.89	3.89
	10										3.89
3	1	4.15	4.08	4.07	4.06	4.06	4.06	4.05	4.05	4.05	4.05
	2		4.00	3.97	3.96	3.96	3.96	3.95	3.95	3.95	3.96
	3			3.95	3.94	3.93	3.92	3.92	3.92	3.92	3.92
	4				3.92	3.91	3.91	3.91	3.90	3.90	3.90
	5					3.91	3.90	3.90	3.90	3.89	3.89
	6						3.89	3.89	3.89	3.88	3.89
	7							3.89	3.89	3.88	3.88
	8								3.88	3.88	3.88
	9									3.88	3.88
	10										3.88
4	1	4.08	4.03	4.01	4.01	4.00	4.00	4.00	4.00	4.00	4.00
	2		3.96	3.94	3.94	3.93	3.93	3.92	3.93	3.93	3.92
	3			3.92	3.91	3.91	3.90	3.90	3.90	3.90	3.90
	4				3.90	3.90	3.89	3.89	3.89	3.89	3.89
	5					3.89	3.89	3.89	3.88	3.88	3.88
	6						3.88	3.88	3.88	3.88	3.87
	7							3.88	3.87	3.87	3.87
	8								3.87	3.87	3.87
	9									3.87	3.87
	10										3.87

Table 5. Exact critical values for $\tilde{R}$ when testing for null hypothesis (15) with $m = 2$, $r \in \{1, \ldots, 4\}$, $s_1 \in \{1, \ldots, 10\}$, $s_2 \in \{s_1, \ldots, 10\}$, and a significance level of 5% (simulation size per value: 2×10^7).

r	$s_1 \setminus s_2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	1.81	2.65	3.11	3.34	3.48	3.58	3.64	3.68	3.73	3.75
	2		2.63	2.77	2.86	3.00	3.13	3.22	3.28	3.34	3.38
	3			3.00	3.08	3.12	3.15	3.17	3.19	3.21	3.23
	4				3.20	3.25	3.28	3.30	3.32	3.32	3.34
	5					3.32	3.35	3.38	3.40	3.41	3.42
	6						3.40	3.43	3.45	3.46	3.47
	7							3.46	3.49	3.50	3.51
	8								3.51	3.53	3.54
	9									3.55	3.56
	10										3.57
2	1	2.63	2.86	3.13	3.28	3.37	3.44	3.49	3.52	3.55	3.58
	2		3.20	3.28	3.32	3.33	3.35	3.36	3.37	3.38	3.39
	3			3.40	3.45	3.47	3.49	3.49	3.50	3.51	3.51
	4				3.51	3.54	3.55	3.57	3.57	3.58	3.58
	5					3.57	3.60	3.61	3.62	3.62	3.63
	6						3.62	3.63	3.64	3.65	3.65
	7							3.65	3.66	3.67	3.67
	8								3.67	3.68	3.69
	9									3.69	3.70
	10										3.71
3	1	3.00	3.15	3.21	3.26	3.31	3.34	3.39	3.42	3.44	3.46
	2		3.40	3.47	3.49	3.50	3.51	3.51	3.52	3.51	3.52
	3			3.54	3.58	3.59	3.61	3.61	3.61	3.61	3.62
	4				3.62	3.64	3.65	3.66	3.66	3.66	3.67
	5					3.66	3.68	3.68	3.69	3.69	3.70
	6						3.69	3.70	3.71	3.71	3.72
	7							3.71	3.72	3.72	3.73
	8								3.73	3.74	3.74
	9									3.74	3.75
	10										3.75
4	1	3.19	3.32	3.35	3.37	3.39	3.40	3.41	3.42	3.43	3.44
	2		3.51	3.56	3.58	3.59	3.59	3.59	3.60	3.59	3.60
	3			3.62	3.65	3.66	3.67	3.66	3.67	3.67	3.67
	4				3.67	3.69	3.70	3.70	3.71	3.71	3.71
	5					3.71	3.72	3.72	3.73	3.73	3.73
	6						3.73	3.73	3.74	3.75	3.75
	7							3.74	3.75	3.76	3.76
	8								3.76	3.76	3.76
	9									3.76	3.77
	10										3.78

4.2. Asymptotic Tests

Finally, we provide asymptotic results for $m = 2$ systems and null hypothesis

$$H_0 : \quad \sigma_1 = \sigma_2 \,. \tag{18}$$

From Formulas (16) and (17), the corresponding test statistics are seen to be

$$\tilde{\Lambda} \quad = \quad 2r \left[s_1 \log \left(\frac{s_1}{s_1 + s_2} \, (1 + 1/\tilde{Q}) \right) + s_2 \log \left(\frac{s_2}{s_1 + s_2} \, (1 + \tilde{Q}) \right) \right] \tag{19}$$

$$\text{and} \quad \tilde{R} \quad = \quad r \left(\frac{1}{s_1} + \frac{1}{s_2} \right) \left(\frac{s_2 \tilde{Q} - s_1}{\tilde{Q} + 1} \right)^2 \tag{20}$$

with statistic $\tilde{Q} = \tilde{T}^{(1)}/\tilde{T}^{(2)}$; cf. Formulas (9) and (10). Note that for $s_1 = s_2$, the likelihood-ratio test and the Rao score test are equivalent, since $\tilde{\Lambda}$ and $\tilde{R}$ are both strictly monotone functions of statistic $(\tilde{T}^{(1)} + \tilde{T}^{(2)})^2/(\tilde{T}^{(1)}\tilde{T}^{(2)})$; see also Section 3.2.

Again, we assume that $s_1/(s_1 + s_2) \to a \in (0,1)$ when the total sample size tends to infinity. Then, by the strong law of large numbers and the central limit theorem,

$$\tilde{Q} \to \frac{a}{1-a}\,\frac{\sigma_2}{\sigma_1} \quad \text{a.s.}\,, \qquad \text{and} \qquad \left(\frac{\tilde{T}^{(1)} + rs_1/\sigma_1}{\sqrt{rs_1}/\sigma_1}, \frac{\tilde{T}^{(2)} + rs_2/\sigma_2}{\sqrt{rs_2}/\sigma_2}\right) \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)\,,$$

which implies that, under null hypothesis (18),

$$\sqrt{\frac{r}{s_1}}(s_2\tilde{Q} - s_1) \xrightarrow{d} \mathcal{N}_1(0, 1/(1-a))$$

(cf. Lemma 1 and its proof). From this, the following theorem can be shown in analogy to the proof of Theorem 2.

Theorem 4. *Under null hypothesis (18), $\tilde{\Lambda}$ and $\tilde{R}$, given by Formulas (19) and (20), are asymptotically $\chi^2(1)$-distributed.*

5. Power Study

We perform a simulation study to investigate and compare the power of the tests derived in Sections 3 and 4, respectively.

For $k = 1,2$, let s_k vectors of r component failure times of some $(n_k - r + 1)$-out-of-n_k system be observed, where $n_1, n_2 \geq r$ are arbitrary integers. In sample $k \in \{1,2\}$, any vector of component failure times is described by the first r $(\leq n_k)$ SOSs based on the cdf $F^{(k)}$ and model parameters $\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}$. Moreover, all $s_1 + s_2$ vectors are assumed to be independent.

First, let $F^{(1)}$ and $F^{(2)}$ be known (but arbitrary), such that the uncertainty of the model is captured within $\alpha_1^{(k)}, \ldots, \alpha_r^{(k)}$, $k = 1,2$, and let $r = 4$. To decide whether both systems are subject to the same load-sharing effects, we consider null hypothesis

$$H_0 : \alpha_j^{(1)} = \alpha_j^{(2)}\,, \quad 1 \leq j \leq 4\,, \tag{21}$$

and apply the exact and asymptotic likelihood-ratio test and Rao score test of Section 3. For a significance level of 5%, different samples sizes $s_1 < s_2$, and three vectors of model parameters describing either no, a linear, or an even faster increase in stress upon failures, Table 6 shows numerical power values at the corresponding pairs.

It is seen from Table 6 that the power of all tests increases when sample sizes increase or the vectors of the model parameters differ more. For small sample sizes, the exact Rao score test turns out to be biased, whereas the exact likelihood-ratio test seems to be unbiased (at least over the alternatives considered). Here, a test is said to be unbiased if its power function, defined on the set of all alternatives, is bounded from below by the significance level of the test; otherwise, the test is called biased. None of the exact tests dominates the other in terms of power. While the power of the likelihood-ratio tests seems to be almost unaffected when sample sizes are interchanged, the Rao score tests have greater power when more observations are recorded from the system with larger load-sharing effects. Moreover, the table indicates that the asymptotic Rao score test is conservative, i.e., its actual level is smaller than the nominal one (of 5%), which implies that the test is biased; on the other hand, the asymptotic likelihood-ratio test seems to be nonconservative and unbiased. Somewhat surprisingly, the actual levels of both asymptotic tests are already close to the nominal one for moderate sample sizes.

Table 6. Power in % at alternatives $\boldsymbol{\alpha}^{(1)} = (\alpha_1^{(1)}, \ldots, \alpha_4^{(1)})$ and $\boldsymbol{\alpha}^{(2)} = (\alpha_1^{(2)}, \ldots, \alpha_4^{(2)})$ of the exact and asymptotic likelihood-ratio test and Rao score test when testing for null hypothesis (21) with samples sizes s_1, s_2 and a significance level of 5% (simulation size per value: 10^6). The last two columns correspond to the asymptotic tests, where numbers in brackets additionally show the actual levels.

$\boldsymbol{\alpha}^{(1)}$	$\boldsymbol{\alpha}^{(2)}$	s_1	s_2	Λ	R	Λ_{as}	R_{as}
(1.0, 1.0, 1.0, 1.0)	(1.1, 1.2, 1.3, 1.4)	3	5	6.7	10.0	8.4 (6.4)	4.6 (1.9)
		13	15	13.5	14.9	14.3 (5.4)	12.1 (3.7)
		21	23	19.8	20.7	20.3 (5.2)	18.5 (4.2)
(1.1, 1.2, 1.3, 1.4)	(1.0, 1.0, 1.0, 1.0)	3	5	6.6	3.5	8.3	1.1
		13	15	13.2	12.0	14.0	9.6
		21	23	19.6	18.8	20.2	16.7
(1.0, 1.0, 1.0, 1.0)	(1.1, 1.3, 1.6, 2.0)	3	5	12.0	18.8	14.4	10.1
		13	15	41.1	43.6	42.3	38.8
		21	23	62.5	63.7	63.2	60.8
(1.1, 1.3, 1.6, 2.0)	(1.0, 1.0, 1.0, 1.0)	3	5	10.9	4.4	13.3	1.3
		13	15	40.3	37.7	41.4	32.7
		21	23	62.0	60.7	62.7	57.6
(1.1, 1.2, 1.3, 1.4)	(1.1, 1.3, 1.6, 2.0)	3	5	6.3	8.8	7.9	3.9
		13	15	11.5	12.5	12.2	10.0
		21	23	16.2	16.8	16.7	14.9
(1.1, 1.3, 1.6, 2.0)	(1.1, 1.2, 1.3, 1.4)	3	5	6.3	4.0	7.8	1.4
		13	15	11.4	10.4	12.0	8.1
		21	23	16.1	15.4	16.6	13.6

Now, suppose that model parameters $\alpha_1^{(k)}, \ldots, \alpha_r^{(k)}$, $k = 1, 2$, are known (but arbitrary), and let baseline cdfs $F^{(1)}$ and $F^{(2)}$ be as stated in Formula (14) with unknown parameters σ_1, σ_2, and known (but arbitrary) functions g_1, g_2. To check for common baseline-distribution parameters, we consider null hypothesis

$$H_0 : \sigma_1 = \sigma_2 , \tag{22}$$

and apply the exact and asymptotic likelihood-ratio test and Rao score test of Section 4. For a significance level of 5% and different values of $s_1 < s_2$ and r, Table 7 shows numerical power values at several alternatives.

From Table 7, it is found that the power of all tests increases with increasing sample sizes, with increasing r, or with growing distance $|\sigma_2 - \sigma_1|$ of the baseline-distribution parameters. Again, the exact and asymptotic Rao score tests turn out to be biased for small sample sizes, and the likelihood-ratio tests seem to be unbiased (over the considered alternatives). None of the exact tests uniformly has greater power than the other. Interchanging sample sizes has an impact on the power of the Rao score tests at a given alternative, whereas the power of the likelihood-ratio tests seems to be nearly invariant. The actual levels of the asymptotic tests are all close to the nominal one (of 5%), where those of the likelihood-ratio test slightly exceed 5% while the Rao score test is conservative.

Table 7. Power in % at alternatives $\sigma = (\sigma_1, \sigma_2)$ of the exact and asymptotic likelihood-ratio test and Rao score test when testing for null hypothesis (22) with samples sizes s_1, s_2 and a significance level of 5% (simulation size per value: 10^6). The last two columns correspond to the asymptotic tests, where the numbers in brackets additionally show the actual levels.

r	σ	s_1	s_2	$\tilde{\Lambda}$	$\tilde{R}$	$\tilde{\Lambda}_{as}$	$\tilde{R}_{as}$
2	(0.9, 1.1)	3	5	6.6	8.2	7.0 (5.4)	6.7 (3.8)
		13	15	11.4	11.9	11.7 (5.1)	11.4 (4.7)
		21	23	15.4	15.7	15.5 (5.1)	15.3 (4.8)
	(0.8, 1.2)	3	5	11.9	15.3	12.6	13.1
		13	15	32.2	33.1	32.6	32.3
		21	23	47.2	47.7	47.4	47.1
	(1.1, 0.9)	3	5	6.5	4.9	7.0	3.5
		13	15	11.3	10.8	11.5	10.4
		21	23	15.4	15.1	15.5	14.7
	(1.2, 0.8)	3	5	11.2	7.7	11.9	5.6
		13	15	31.8	30.8	32.2	30.0
		21	23	46.9	46.4	47.1	45.7
3	(0.9, 1.1)	3	5	7.5	9.1	7.8 (5.3)	8.0 (4.2)
		13	15	14.9	15.4	15.0 (5.1)	14.9 (4.8)
		21	23	21.0	21.3	21.0 (5.1)	20.9 (4.9)
	(0.8, 1.2)	3	5	15.8	19.1	16.3	17.4
		13	15	45.5	46.3	45.7	45.6
		21	23	64.1	64.5	64.2	64.0
	(1.1, 0.9)	3	5	7.4	5.8	7.7	4.8
		13	15	14.9	14.4	15.0	14.0
		21	23	20.9	20.6	21.0	20.2
	(1.2, 0.8)	3	5	15.0	11.5	15.4	9.7
		13	15	45.1	44.3	45.3	43.5
		21	23	63.9	63.5	63.9	63.0
4	(0.9, 1.1)	3	5	8.4	10.0	8.7 (5.2)	9.2 (4.4)
		13	15	18.2	18.7	18.5 (5.1)	18.5 (4.9)
		21	23	26.4	26.7	26.5 (5.0)	26.4 (4.9)
	(0.8, 1.2)	3	5	19.6	22.9	20.1	21.6
		13	15	56.6	57.3	57.0	57.1
		21	23	76.5	76.7	76.5	76.5
	(1.1, 0.9)	3	5	8.2	6.6	8.5	5.9
		13	15	18.1	17.6	18.4	17.4
		21	23	26.3	25.9	26.3	25.6
	(1.2, 0.8)	3	5	18.6	15.1	19.1	13.7
		13	15	56.3	55.5	56.7	55.3
		21	23	76.3	76.0	76.4	75.7

6. Conclusions

In a setup of multiple samples of sequential order statistics modelling the component lifetimes of possibly differently structured k-out-of-n systems, we provided exact and asymptotic statistical tests with flexible hypotheses to check for common load-sharing parameters as well as for common baseline-distribution parameters. The corresponding test statistics are shown to have single null distributions, i.e., they each have only one distribution under all parameters specified by the null hypothesis, such that exact critical values subject to a desired significance level are readily obtained by using Monte Carlo simulations. The proposed tests can also be used to decide whether a meta-analysis of the underlying data is reasonable. If, based on some dataset, the null hypothesis of common load-sharing parameters (or common baseline-distribution parameters) is not rejected, the performance of statistical procedures as, for instance, the accuracy of estimators, may be increased when applied to the whole dataset; this is, in particular, relevant for small sample sizes that are prevalent in reliability.

Finally, the derived results might also be useful in other reliability applications. On the one hand, by appropriately setting the model parameters, the presented tests may be applied to check for identical scale parameters of underlying lifetime distributions in differently designed progressively type-II censored lifetime experiments (see, e.g., Reference [28]). On the other hand, by choosing a standard exponential baseline distribution, we may test for equality of parameters associated with stress levels in multiple repeated type-II censored exponential step-stress experiments (see Reference [29]).

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Abbreviations

a.s.	almost surely
cdf	cumulative distribution function
iid	independent and identically distributed
MLE	maximum-likelihood estimator
SOS	sequential order statistic
wrt	with respect to

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