

Article

A Stochastic Poisson Structure

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Received: 2 July 2009; in revised form: 26 July 2009 / Accepted: 12 August 2009 /

Published: 20 August 2009

Abstract: We define a Poisson structure on the Nualart-Pardoux test algebra associated to the path space of a finite dimensional Lie algebra.

Keywords: Poisson structure; Malliavin Calculus

1. Introduction

Deformation quantization in field theory is an interesting subject [1–3]. Motivated by that, Dito ([4]) has considered the canonical Poisson structure on $\mathbb{H} \oplus \mathbb{H}^*$ where \mathbb{H} is a Hilbert space. Dito has defined the Moyal product on the algebra of Hilbert-Schmitt smooth functions on $\mathbb{H} \oplus \mathbb{H}^*$.

Models of stochastic analysis, if they are similar to models of quantum field theory, are more tractable. As a consequence, infinite dimensional analysis, although very similar to highly studied objects of quantum field theory, is much more easier to handle.

There are more or less two types of infinite dimensional analysis:

-)The Malliavin Calculus, which is analytical.
-)White noise analysis, which is algebraic.

Dito-Léandre [5] has interpreted [4] inside the Malliavin Calculus, the algebra being the Malliavin test algebra of functionals smooth in the Malliavin sense on the Wiener space.

Léandre has developed the paper [5] by considering more regular symplectic structure which leads to the study of more singular Poisson structure in [6–9], by considering the Hida test algebra endowed with a normalized Wick product instead of the Malliavin test algebra endowed with the Wiener product. The main difference between white noise analysis and the Malliavin Calculus is that the Hida test algebra is much more smaller than the Malliavin test algebra. The space of allowed operations in white noise analysis is much more bigger than the space of allowed operations in the Malliavin Calculus.

For a review of deformation quantization in infinite dimensional analysis, we refer to the review of Léandre [7].

There are a lot of works on Poisson structure in infinite dimension [10], in part motivated by the theory of integrable systems. We refer to the review of Mokhov [11] and Dubrovin-Novikov [12] for that.

We would like to give an analytical meaning to the simplest Poisson structure considered in the review of Dubrovin-Novikov [12] formula (44) p 53). Let us recall how it works. We consider the set of paths $t \rightarrow x(t)$ from $[0, 1]$ into \mathbb{R}^m . The Poisson structure is defined by

$$\{x_i(s), x_j(t)\} = \delta_0(s - t) \sum_k c_{i,j}^k x_k(t) \quad (1)$$

$c_{i,j}^k$ are the structural coefficient of a Lie algebra structure on $(\mathbb{R}^m)^*$; δ_0 is the Dirac mass in 0.

In white noise analysis, people replace the Poisson bracket $\{, \}$ by a Lie bracket $[,]$, in order to get a representation of the canonical commutation relations on the Hida Fock space. The white noise $x(t)$ is classically represented as $a(t) + a(t)^*$, where

$$[a(t), a(s)] = [a^*(s), a^*(t)] = 0 \quad (2)$$

$$[a(t), a^*(s)] = \delta_0(s - t) \quad (3)$$

So we have to identify $x(t)$ as a white noise in (1) and not as a Brownian motion, in order to understand analytically the Poisson structure (1). But this Poisson structure will lead to the introduction of anticipative Stratonovich integrals. The Malliavin test algebra is not adequate to understand anticipative Stratonovich integrals.

Nualart-Pardoux pioneered the study of anticipative Stratonovich integrals in [13]. Functional spaces were suitably defined by Léandre [14–18], in order to understand anticipative Stratonovich integrals in a convenient way.

Léandre has defined a Nualart-Pardoux Calculus, which is a refinement of the Malliavin Calculus and allows to define various stochastic cohomology theories on various loop spaces ([14–18]). The Nualart-Pardoux test algebra $N.P_{\infty-}$ is smaller than the Malliavin test algebra for the Brownian motion on \mathbb{R}^m . But the anticipative Statonovich integrals act continuously on $N.P_{\infty-}$. We get the main theorem of this work:

Theorem 1 *The Poisson structure $\{, \}$ in (1) acts continuously on $N.P_{\infty-}$*

2. A Brief Review on the Nualart-Pardoux Calculus

We consider the set of continuous paths $C([0, 1]; \mathbb{R}^m)$ from $[0, 1]$ into \mathbb{R}^m endowed with the uniform topology. A typical path is denoted by $t \rightarrow B(t) = (B_i(t))$, on which we consider the Brownian motion measure dP ([19]).

Let us recall how we construct dP . We consider the Cameron-Martin Hilbert space H [20–21] of maps from $[0, 1]$ into \mathbb{R}^m such that

$$\int_0^1 |h'_s|^2 ds = \|h\|^2 < \infty \quad (4)$$

dP is formally the Gaussian probability measure

$$1/Z \exp[-\|h\|^2/2]dD(h) \quad (5)$$

dD is the formal Lebesgue measure on H which does not exist as a measure (We refer to the works of Léandre [22–25], Asada [26], Pickrell [27] for various approaches to the Lebesgue measure in infinite dimension). Let H_1 be a finite dimensional real Hilbert space ($h_1 \in H_1$) with Hilbert norm $\|\cdot\|_1$. Let us consider the centered normalized Gaussian measure on H_1 . It is classically represented by $\sum e_i N_i$ where N_i are centered normalized independent one dimensional Gaussian variables and the system of e_i constitutes an orthonormal basis of the real Hilbert space H_1 .

It should be tempting to represent dP by using the same procedure. We consider an orthonormal basis e_i of H . The law of the Brownian motion is represented by the series $\sum e_i N_i$ where the N_i is a collection of independent centered one dimensional Gaussian variables. This series **does not converge** in H but in $C([0, 1]; \mathbb{R}^m)$ [28]. We refer to the textbook of Kuo [29] for the theory of infinite dimensional Gaussian measures.

Let us consider a functional F on $C([0, 1]; \mathbb{R}^m)$. Its r^{th} stochastic derivative $d^r F$, according to the framework of the Malliavin Calculus [30–32], is defined if it exists by

$$d^r F(h_1, \dots, h_r) = \int_{[0,1]^r} \langle d^r F(s_1, \dots, s_r), h'_{1,s_1}, \dots, h'_{r,s_r} \rangle ds_1 \dots ds_r \quad (6)$$

where h_i belongs to the Hilbert space of paths H from $[0, 1]$ into \mathbb{R}^m such that

$$\int_0^1 |h'_s|^2 ds = \|h\|^2 < \infty \quad (7)$$

The Sobolev norms of the Malliavin Calculus are defined by the following formula. If F is a Brownian functional,

$$E\left[\left(\int_{[0,1]^r} |d^r F(s_1, \dots, s_r)|^2 ds_1 \dots ds_r\right)^{p/2}\right]^{1/p} = \|F\|_{r,p} \quad (8)$$

The Malliavin test algebra is consisted of Brownian functionals F such that for all r, p , $\|F\|_{r,p} < \infty$. Let us recall how we construct these Sobolev spaces. Let f be a smooth function from $(\mathbb{R}^m)^d$ into \mathbb{R} with compact support and some times $0 < t_1 < \dots < t_d \leq 1$. We introduce the cylindrical functional $F(B(\cdot)) = f(B(t_1), \dots, B(t_d))$. We consider the Gateaux derivative of F along a deterministic direction h of H :

$$\langle dF, h \rangle = \left\langle \sum \frac{\partial}{\partial x_i} f(B(t_1), \dots, B(t_d)), h_{t_i} \right\rangle \quad (9)$$

There is absolutely no problem to define it. We have the integration by parts formula true for any cylindrical functional

$$E[\langle dF, h \rangle] = E\left[F \int_0^1 \langle h'_s, \delta B(s) \rangle\right] \quad (10)$$

where $\delta B(s)$ is the Itô differential. The Itô integral is the limit in all the $L^p(dP)$ $p < \infty$ of the sum $\sum \langle h'_{s_i}, B(s_{i+1}) - B(s_i) \rangle$ where $0 < s_1 < \dots < s_i < s_{i+1} < \dots < s_{2^n-1} < 1 = s_{2^n}$ is a dyadic subdivision of $[0, 1]$ of length 2^{-n} . The convergence does not pose any problem because h is deterministic. Since we have the integration by parts formula (10), we can extend the operation of taking the stochastic

derivative of a Brownian functional F consistently, as we establish classically the definition of Sobolev spaces in finite dimension. The main novelty of the Malliavin Calculus with respects of its precursors motivated by mathematical physics [33–36] is that the **algebra** of functionals which belong to all the Sobolev spaces of the Malliavin Calculus (these functionals are said to be smooth in the Malliavin sense) is constituted of functionals almost surely defined. The reader interested in the Malliavin Calculus can see the books [30,32].

If we consider the same dyadic subdivision as before, we can introduce the polygonal approximation $B^n(t)$ of $B(t)$. Let us consider a (non deterministic!) map from $[0, 1]$ into \mathbb{R}^m $t \rightarrow h_t$ which belongs to $L^2([0, 1]; \mathbb{R}^m)$. We can consider the random ordinary integral $\int_0^1 \langle h_t, dB^n(t) \rangle$. It has no problem to be defined. To pass to the limit when $n \rightarrow \infty$, there are a lot of problems. If we can pass to the limit, we say that the limit $\int_0^1 \langle h_t, dB(t) \rangle$ is an anticipative Stratonovich integral. Nualart-Pardoux [13] are the first authors who have defined some anticipative Stratonovich integrals. A convenient theory was established by Léandre [14–18] in order to understand some Sobolev cohomology theories on the loop space. Let us recall it quickly.

We consider another set of Sobolev norms ([14]). We suppose that outside the diagonals of $[0, 1]^r$

$$E[|d^r F(s_1, \dots, s_r) - d^r F(s'_1, \dots, s'_r)|^p]^{1/p} \leq C_{r,p} \sum |s_i - s'_i|^{1/2} \quad (11)$$

The smallest $C_{r,p}$ such that the previous inequality is satisfied is called the first Nualart-Pardoux Sobolev norm. The second Nualart-Pardoux Sobolev norm is the smallest $C_{r,p}^1$ such that for all $(s_i) \in [0, 1]^r$,

$$E[|d^r F(s_1, \dots, s_r)|^p]^{1/p} \leq C_{r,p}^1 \quad (12)$$

Definition 2 *The Nualart-Pardoux test algebra $N.P_{\infty-}$ is consisted of functionals F whose all Nualart-Pardoux Sobolev norms of first type and second type are finite. Elements F of $N.P_{\infty-}$ are said to be smooth in the Nualart-Pardoux sense.*

Let us recall that $N.P_{\infty-}$ is an algebra [14].

We can consider a random element of $L^2([0, 1]; \mathbb{R}^m)$ $t \rightarrow h_t$. We can consider its r^{th} stochastic derivative

$$d^r h_t(h_1, \dots, h_r) = \int_{[0,1]^r} \langle d^r h_t(s_1, \dots, s_r), h'_{1,s_1}, \dots, h'_{r,s_r} \rangle ds_1 \dots ds_r \quad (13)$$

Its first Nualart-Pardoux Sobolev norm $C_{r,p}$ is the smallest number such that outside the diagonals of $[0, 1] \times [0, 1]^r$

$$E[|d^r h_t(s_1, \dots, s_r) - d^r h_{t'}(s'_1, \dots, s'_r)|^p]^{1/p} \leq C_{r,p} (|t - t'|^{1/2} + \sum |s_i - s'_i|^{1/2}) \quad (14)$$

The second type of Nualart-Pardoux Sobolev norm $C_{r,p}^1$ of $h(\cdot)$ is the smallest number such that for all $(t, s_1, \dots, s_r) \in [0, 1] \times [0, 1]^r$

$$E[|d^r h_t(s_1, \dots, s_r)|^p]^{1/p} \leq C_{r,p}^1 \quad (15)$$

Let us recall the theorem of Léandre [14]:

Theorem 3 Let h be a random element of $L^2([0, 1]; \mathbb{R}^m)$ such that all its Nualart-Pardoux Sobolev norm are finite. Then the anticipative Stratonovich integral

$$\int_0^1 \langle h_t, dB(t) \rangle \quad (16)$$

is smooth in the Nualart-Pardoux sense and its Nualart-Pardoux Sobolev norms can be estimated in terms of the Nualart-Pardoux norms of H .

In such a case, $\int_0^1 \langle h_t, dB(t) \rangle$ is the limit in all the $L^p(dP)$ $p < \infty$ of $\int_0^1 \langle h_t, dB^n(t) \rangle$.

Moreover,

$$\langle d(\int_0^1 \langle h_t, dB(t) \rangle), \tilde{h} \rangle = \int_0^1 \langle \langle dh_t, \tilde{h} \rangle, dB(t) \rangle + \int_0^1 \langle h_t, d/dt \tilde{h}_t \rangle dt \quad (17)$$

This means that the kernel of the stochastic derivative of $\int_0^1 \langle h_t, dB(t) \rangle$ is $\int_0^1 \langle dh_t(s), dB(t) \rangle + h_s$.

Let us explain this formula: in order to take the stochastic derivative of $\int_0^1 \langle h_t, dB(t) \rangle$, we do the same formal computations as if the anticipative Stratonovich integral had been a classical integral: we take first of all derivatives of h_t which lead to the term $\langle dh_t, \tilde{h} \rangle$ and derivative of dB_t which lead to $d/dt \tilde{h}_t dt$.

3. The Poisson Structure on the Nualart-Pardoux Test Algebra

Let us recall what is a Poisson bracket $\{, \}$. We consider a commutative Frechet unital real algebra endowed with a family of Banach norms $\|\cdot\|_p$. This means that for all p , there exists p^1 such that for all F^1, F^2 in A

$$\|F^1 F^2\|_p \leq C_p \|F^1\|_{p^1} \|F^2\|_{p^1} \quad (18)$$

A Poisson Bracket is a bilinear map from $A \times A$ into A , which is a derivation in each argument, vanishes on the unit. The derivation property means that for all F^1, F^2, F^3 in A

$$\{F^1 F^2, F^3\} = F^1 \{F^2, F^3\} + \{F^1, F^3\} F^2 \quad (19)$$

Moreover, it satisfies the following properties: If F^1, F^2, F^3 belong to A ,

$$\{F^1, F^2\} = -\{F^2, F^1\} \quad (20)$$

$$\{\{F^1, F^2\}, F^3\} + \{\{F^2, F^3\}, F^1\} + \{\{F^3, F^1\}, F^2\} = 0 \quad (21)$$

Moreover, for all p , there exists p' such that

$$\|\{F^1, F^2\}\|_p \leq C \|F^1\|_{p'} \|F^2\|_{p'} \quad (22)$$

In the sequel, we will choose $A = N.P_{\infty-}$. We consider the structural constants $c_{i,j}^k$ of a Lie algebra structure on $(\mathbb{R}^m)^*$. The stochastic gradient dF of a functional F can be written $dF = (dF_i)$. Formula (1) reads in this framework

$$\{F^1, F^2\} = \sum_{i,j,k} \int_0^1 dF_i^1(s) dF_j^2(s) c_{i,j}^k dB_k(s) \quad (23)$$

where we consider a Stratonovitch anticipative integral.

Proof of Theorem 1 In order to show that (19) is satisfied, we remark that $d(F^1 F^2) = F^1 dF^2 + F^2 dF^1$. Moreover, since we consider anticipative Stratonovich integrals, we have $\int_0^1 \langle gh_t, dB_t \rangle = g \int_0^1 \langle h_t, dB_t \rangle$ almost surely. Then (19) is satisfied.

Only the Jacobi relation (21) put a problem. We have:

$$d_{i^4} \{F^1, F^2\}(t) = \sum_{i^1, i^2, k} \int_0^1 d_{i^1, i^4} F^1(s, t) d_{i^2} F^2(s) c_{i^1, i^2}^k dB_k(s) \\ + \sum_{i^1, i^2, k} \int_0^1 d_{i^2, i^4} F^2(s, t) d_{i^1} F^1(s) c_{i^1, i^2}^k dB_k(s) + \sum_{i^1, i^2} d_{i^1} F^1(t) d_{i^2} F^2(t) c_{i^1, i^2}^{i^4} \quad (24)$$

Therefore

$$\{\{F^1, F^2\}, F^3\} = \sum_{i^1, i^2, i^3, i^4, k, k'} \int_{[0,1]^2} d_{i^1, i^4} F^1(s, t) d_{i^2} F^2(s) d_{i^3} F^3(t) c_{i^1, i^2}^k c_{i^4, i^3}^{k'} dB_k(s) dB_{k'}(t) \\ + \sum_{i^1, i^2, i^3, i^4, k, k'} \int_{[0,1]^2} d_{i^1} F^1(s) d_{i^2, i^4} F^2(s, t) d_{i^3} F^3(t) c_{i^1, i^2}^k c_{i^4, i^3}^{k'} dB_k(s) dB_{k'}(t) \\ + \sum_{i^1, i^2, i^3, i^4, k} \int_0^1 d_{i^1} F^1(s) d_{i^2} F^2(s) d_{i^3} F^3(s) c_{i^1, i^2}^{i^4} c_{i^4, i^3}^k dB_k(s) = A_{1,2,3} + B_{1,2,3} \quad (25)$$

If e_i is the canonical basis of $(\mathbb{R}^m)^*$, we have

$$[e_i, e_j] = \sum c_{i,j}^k e_k \quad (26)$$

such that the Jacobi relation

$$[[e_{i^1}, e_{i^2}], e_{i^3}] + [[e_{i^2}, e_{i^3}], e_{i^1}] + [[e_{i^3}, e_{i^1}], e_{i^2}] = 0 \quad (27)$$

reads

$$\sum_{k'} c_{i^1, i^2}^{k'} c_{k', i^3}^k + \sum_{k'} c_{i^2, i^3}^{k'} c_{k', i^1}^k + \sum_{k'} c_{i^3, i^1}^{k'} c_{k', i^2}^k = 0 \quad (28)$$

Therefore,

$$B_{1,2,3} + B_{2,3,1} + B_{3,1,2} = 0 \quad (29)$$

Let us show that

$$A_{1,2,3} + A_{2,3,1} + A_{3,1,2} = 0 \quad (30)$$

The second derivative of F^1 appears only in $\{\{F^1, F^2\}, F^3\}$ and in $\{\{F^3, F^1\}, F^2\}$. More precisely $d_{i^1, i^4} F^1(s, t)$ appears in

$$\sum_{i^2, i^3, k, k'} \int_{[0,1]^2} d_{i^1, i^4} F^1(s, t) d_{i^2} F^2(s) d_{i^3} F^3(t) c_{i^1, i^2}^k c_{i^4, i^3}^{k'} dB_k(s) dB_{k'}(t) \\ + \sum_{i^2, i^3, k, k'} \int_{[0,1]^2} d_{i^3} F^3(s) d_{i^1, i^4} F^1(s, t) d_{i^2} F^2(t) c_{i^3, i^1}^k c_{i^4, i^2}^{k'} dB_k(s) dB_{k'}(t) = A_{i^1, i^4} \quad (31)$$

But by the Schwarz relation

$$d_{i^1 i^4} F^1(s, t) = d_{i^4 i^1} F^1(t, s) \quad (32)$$

Moreover $[e_i, e_j] = -[e_j, e_i]$ such that

$$c_{i^3, i^1}^k = -c_{i^1, i^3}^k \quad (33)$$

This shows that

$$\sum_{i^1, i^4} A_{i^1, i^4} = 0 \quad (34)$$

Acknowledgments

We thank B. Dubrovin and V. Rubtsov for helpful discussions.

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