Symmetry **2010**, *2*, 98-111; doi:10.3390/sym2010098

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Symmetries and (Related) Recursion Operators of Linear Evolution Equations

Giampaolo Cicogna

Dipartimento di Fisica "E. Fermi" dell'Università di Pisa and INFN, sez. di Pisa, Largo B. Pontecorvo 3, Ed. B-C, I-56127–Pisa, Italy; E-Mail: cicogna@df.unipi.it

Received: 28 December 2009; in revised form: 26 January 2010 / Accepted: 29 January 2010 / Published: 5 February 2010

Abstract: Significant cases of time-evolution equations, the linear Schrödinger and the Fokker–Planck equation are considered. It is known that equations of this type can be transformed, in some cases, into a highly simplified form. The properties of these equations in their initial and their simplified form are compared, showing in particular that this transformation partially prevents a clear understanding and a full application of the (physically relevant) notion of the so-called step up/down operators. These operators are shown to be recursion operators, related to the Lie point symmetries of the equations, which are also carefully discussed.

Keywords: recursion operators; step up/down operators; Lie point symmetries; Schrödinger equation; Fokker–Planck equation

1. Introduction

In this paper we examine in detail some significant cases of two linear evolution equations, namely the linear Schrödinger equation (Sect. 3) and the Fokker–Planck equation (Sect. 4), which can be considered as the prototypes of linear time-evolution partial differential equations (PDE). It is known that, by means of suitable transformations of both the independent and the dependent variables, equations of this type can be put, in some cases, into a highly simplified form; we will discuss this point, comparing the properties of the equations in their initial and in their simplified form. Although the latter form is clearly useful to find, e.g., the Lie point symmetries and the solutions as well, we will show that transforming the equations into simplified form partially prevents and in some sense "hides" a clear understanding and a full application of the notion of the so-called step up/down operators. These "ladder" operators not only produce families of solutions of higher and higher degree (starting e.g. from some elementary solution) but also possess a neat physical interpretation: they indeed connect different solutions with a well-defined behaviour, by increasing/decreasing by a fixed amount some characterizing physical property (e.g., the energy in the case of Schrödinger equation, the time-decay rate in the case of Fokker–Planck one).

The step up/down operators are a simple class of linear recursion operators, related to the symmetries of the differential equations: their relevant properties will be preliminarily presented in Sect. 2.

In this paper we will be concerned only with the usual notion of Lie point symmetries; we refer to standard books (e.g., [1–6]) for any detail about their properties and their determination. We will denote by X (or Y) the infinitesimal generators of such a symmetry; with a little but commonly accepted abuse of language, we will call X (or Y) both the symmetry and its generator.

2. The recursion and the step up/down operators

First of all, let us point out some general results, which will be particularly useful in the following.

Let $u = u(x_i)$ be a function of the p real variables x_i, and X a vector field of the form (sum over i)

$$X \ = \ \xi_i(x)\frac{\partial}{\partial x_i} + B(x)u\frac{\partial}{\partial u}\ ;$$

it is very convenient to introduce the linear differential operator $\mathcal{Q}$ associated to X defined by

$$\mathcal{Q} := -\xi_i\, D_{x_i} + B \tag{1}$$

where D_{x_i} denotes the total derivative. Notice that, putting

$$Q := \mathcal{Q}(u) \ = \ -\xi_i u_{x_i} + B\, u$$

the operator X_Q defined by

$$X_Q := Q\frac{\partial}{\partial u}$$

is the so-called *evolutionary operator*, related to the above vector field X and essentially equivalent to it, as is well known [3], meaning that X is a symmetry for a differential equation $\Delta = \Delta(x_i, u, u_{x_i}, \ldots) = 0$ if and only if the same is true for X_Q.

We then have [3]:

Proposition 1 *Let $\Delta = 0$ be a linear PDE and X one of its symmetries; then the operator $\mathcal{Q}$ is a "recursion operator": i.e., given any symmetry X_0, with its associated operator $\mathcal{Q}_0$, then also*

$$X_{Q_1} := Q_1\frac{\partial}{\partial u} \quad where \quad Q_1 := \mathcal{Q}\big(\mathcal{Q}_0(u)\big)$$

is a symmetry for the PDE.

Some relevant properties of the operator $\mathcal{Q}$ are described by the following (see also [3,6]):

Proposition 2 *Let $\Delta = 0$ be a differential equation and X one of its symmetries; then looking for the solution w_0 to $\Delta = 0$ which is left invariant by X amounts to solve the equation*

$$\mathcal{Q}(w_0) \;=\; 0$$

where $\mathcal{Q}$ is the operator defined in (1). If $\Delta = 0$ is linear and u_0 is any of its solutions, then also

$$u_1 := \mathcal{Q}(u_0)\,, \quad \ldots\,, \quad u_{n+1} := \mathcal{Q}(u_n)$$

for all $n = 1, 2, \ldots$, solve the equation $\Delta = 0$. If X_0 is another symmetry for $\Delta = 0$, then also

$$Q_1(u_0) = \mathcal{Q}\big(\mathcal{Q}_0(u_0)\big)$$

solves the equation, and so on.

Proof. The first statement is standard. For the second part, let us write the linear differential equation in the form

$$\Delta \;=\; \sum_J \alpha_J(x) D_J u \;=\; 0$$

where D_J stand for all possible differentiations of u. If X is any symmetry for this equation, then $X^*\Delta|_{\Delta=0} = 0$ where X^* is the appropriate prolongation of X; on the other hand, recalling that the prolongation of $Q = \mathcal{Q}(u)$ is $D_J Q$, this becomes

$$X^*\Delta|_{\Delta=0} \;=\; \sum_J \alpha_J(x) D_J \mathcal{Q}(u)|_{\Delta=0} \;=\; 0$$

which expresses just the fact that $\mathcal{Q}(u)$ solves the equation whenever u is a solution. The final assertion follows from Proposition 1. □

The notion of ladder or step up/down operators (also called creation/annihilation operators) is contained in the following Proposition.

Proposition 3 *Let A be a linear operator having an eigenfunction u with eigenvalue σ; a linear operator S satisfying*

$$[\,A\,, S\,] \;=\; \kappa\, S$$

where κ is a constant, is a step up/down operator for A, meaning that

$$A(S\,u) \;=\; (\sigma + \kappa)(S\,u)\;.$$

Indeed, $A(Su) = S(Au) + \kappa S u = (\sigma + \kappa)(S\,u)$. □

The idea of introducing various classes of recursion operators has been often and successfully used in the study of differential equations in many different contexts. We refer to [3] for a general introduction; recursion operators of pseudodifferential or integro-differential type have been applied e.g. to nonlinear evolution equations (see [7–9] and references therein); a different hierarchy of operators related to nonclassical symmetries has been proposed in [10] and references therein. Much more simply, the recursion operators considered in this paper are linear operators directly related to the classical Lie point symmetries of the equations, and more specifically to the physically relevant notion of step up/down operators.

3. The Schrödinger equation

We start considering the parabolic partial differential equation for $u = u(x, t)$

$$u_t = u_{xx} + W(x)\, u \tag{2}$$

where $W(x)$ is a given smooth function. For this equation, Ovsjannikov [1] has shown the following result.

Theorem 4 *Apart from the infinite-dimensional algebra of symmetries $X_w = w\,\partial/\partial u$, where $w = w(x,t)$ is any solution to (2), eq. (2) admits a 6-dimensional algebra of Lie point symmetries which is generated by the two trivial symmetries*

$$X_t = \frac{\partial}{\partial t}\,, \quad X_u = u\frac{\partial}{\partial u} \tag{3}$$

and by 4 other linearly independent symmetries, if and only if $W(x)$ has the form

$$W(x) = \alpha x^2 + \beta x + \gamma \tag{4}$$

($\alpha\,,\beta\,,\gamma$ = const.); in this case there is a point transformation

$$x \to y = y(x,t)\,, \qquad t \to s = s(x,t)\,, \qquad u \to v = \omega(x,t)u \tag{5}$$

such that in these variables eq. (2) takes the form of the heat equation

$$v_s = v_{yy}\,. \tag{6}$$

Eq. (2) admits instead, in addition to the trivial ones as above, 2 linearly independent symmetries if and only if

$$W(x) = \alpha x^2 + \beta x + \gamma + \frac{\delta}{(x + x_0)^2} \tag{7}$$

where $\delta = \mathrm{const} \neq 0$, and in this case it can be transformed into an equation where $\alpha = \beta = \gamma = x_0 = 0$. No nontrivial symmetry is admitted for other $W(x)$.

This result can be deduced by means of explicit calculations based on the symmetry determining equations; the symmetry analysis of equations of the form (2) can be also performed using a direct "geometric" approach valid for more general quasi-linear PDE's [11,12].

Notice that, as shown in [13], for any point or contact transformation between two evolution equations, the transformation component for t depends only on t; then, it is enough to consider in eq. (5) only $t \to s = s(t)$. We refer to [13] for a full discussion about point transformations of the very general form $y = y(x,t,u), s = s(x,t,u), v = v(x,t,u)$, including also discrete symmetries, which connect a wide class of linear and nonlinear PDE's.

3.1. The Schrödinger equation for the harmonic oscillator and for the free particle

We are here interested in the 1-dimensional linear Schrödinger equation for a quantum particle moving in a given potential $V = V(x)$, *i.e.*

$$i\, u_t = -\frac{1}{2}u_{xx} + V(x)u\,.$$

The extension of Theorem 4 to cover this equation requires some care for what concerns the symmetry properties, due to the presence of the imaginary factor i, as we shall see; but it is easily seen that this theorem implies that if $V(x)$ is as $W(x)$ in (4), which is the case of the quantum harmonic oscillator (let us assume $\alpha > 0$), then the equation can be transformed into the Schrödinger equation for the *free* particle (*i.e.* $V = 0$)

$$i\,v_s \;=\; -\frac{1}{2}v_{yy}\;. \tag{8}$$

We shall first consider in detail this special case.

Following again Ovsjannikov results, we can give the explicit expression of the needed transformation (5) in this case: first of all, it is not restrictive to put in (4) $\beta = \gamma = 0$ $\big($these constants can be reabsorbed by a translation of x and changing u into $u\exp(i\gamma t)\big)$, let also $\alpha = 1/2$; then, the Schrödinger equation

$$i\,u_t \;=\; -\frac{1}{2}u_{xx} + \frac{1}{2}x^2\,u \tag{9}$$

is transformed into eq. (8) by means of the transformation (involving the imaginary factor i)

$$x \to y = \exp(-it)\,x\,,\quad t \to s = \frac{i}{2}\exp(-2it)\,,\quad u \to v = \exp(x^2/2)\exp(it/2)u(x,t)\;. \tag{10}$$

However, despite this connection among the two equations (cf. also [14,15]), the physical properties of the solutions of the free Schrödinger equation (8) are strongly different from those of the harmonic oscillator (9). For instance, the spectrum of the energy operator $i\partial/\partial t$ is continuous for the free particle, in contrast with the case of the harmonic oscillator; in addition, the solutions which are eigenfunctions of $i\partial/\partial t$ are not normalizable $\big($*i.e.* $\notin L^2(\mathbf{R})\big)$ in the first case, differently from the other one, and so on. So, some care must be used also when transforming solutions from one to the other equation.

Incidentally, also the transformation

$$x \to y = \exp(it)\,x\,,\quad t \to s = -\frac{i}{2}\exp(2it)\,,\quad u \to v = \exp(-x^2/2)\exp(-it/2)u(x,t)$$

does map eq. (9) into (8), but this is scarcely useful: it transforms indeed, for instance, the obvious solution $v(s,y) = 1$ of (8) into the divergent solution $u = \exp(x^2/2)\exp(it/2)$ of (9), with no interest from the physical point of view.

It is particularly important, instead, to anticipate (this point will be considered in detail in the following) that the solution $v = 1$ is transformed by (10) into

$$u_0(x,t) \;=\; \exp(-x^2/2)\exp(-it/2)$$

which is just the solution of (9) describing the first stationary state (with the lowest eigenvalue of the energy $E_0 = 1/2$) of the quantum harmonic oscillator.

Apart from the infinite-dimensional algebra $Y_w = w\,\partial/\partial v$ as in the general case (2), the algebra of symmetries admitted by eq. (8) becomes in this case 7-*dimensional*: it is generated by the trivial symmetries

$$Y_v \;=\; v\frac{\partial}{\partial v}\;,\quad Y_v' \;=\; i\,v\frac{\partial}{\partial v}\;,\quad Y_s \;=\; \frac{\partial}{\partial s} \tag{11}$$

and by these 4 other (linearly independent) symmetries

$$Y_1 = \frac{\partial}{\partial y}, \qquad Y_2 = s\frac{\partial}{\partial y} + y\Big(i\,v\frac{\partial}{\partial v}\Big), \qquad Y_3 = y\frac{\partial}{\partial y} + 2s\frac{\partial}{\partial s}, \\ Y_4 = 2sy\frac{\partial}{\partial y} + 2s^2\frac{\partial}{\partial s} + (iy^2 - s)v\frac{\partial}{\partial v}. \tag{12}$$

We are now using operators and transformations involving simultaneously real and complex quantities. This would require in principle to introduce a suitable prolongation of the equations to the complex plane. Actually, the operators (11-12) are real in their parts associated with the independent variables, and this guarantees that they remain real under the transformations generated by these operators. Notice also that the two generators Y_v and Y'_v in (11) are to be considered as different operators, indeed under a finite transformation with a real parameter they generate respectively a multiplication by a real coefficient and by a phase factor. On the other hand, this is a common situation which is met whenever one deals with symmetries of (linear and nonlinear) Schrödinger equation: see e.g. [6,14–18]; see also [19] for a full discussion about the symmetry properties of the equations in Quantum Mechanics.

It can be interesting to compare the above symmetries of the free linear Schrödinger equation with the nonlinear cubic version

$$i\,v_s + \frac{1}{2}v_{yy} + |v|^2 v = 0\,. \tag{13}$$

As shown in [18], the algebra of symmetries of this equation is generated by (in our notations)

$$\mathcal{Y}_s = \frac{\partial}{\partial s}, \quad \mathcal{Y}'_v = iv\frac{\partial}{\partial v}, \quad \mathcal{Y}_1 = \frac{\partial}{\partial y}, \quad \mathcal{Y}_2 = s\frac{\partial}{\partial y} + y\Big(i\,v\frac{\partial}{\partial v}\Big), \quad \mathcal{Y}_3 = y\frac{\partial}{\partial y} + 2s\frac{\partial}{\partial s} - v\frac{\partial}{\partial v}\,.$$

Not surprisingly, the generators Y_s, Y'_v, Y_1, Y_2 survive, whereas Y_v is lost, and $Y_3\,, Y_4$ are replaced by $\mathcal{Y}_3$.

Coming back to our free Schrödinger equation (8), we now apply to it the ideas and the results presented in Sect. 2. We start from the solution $v = v_0(y, s) = 1$, which is clearly invariant under the symmetry operator Y_1; if one applies repeatedly to this solution the operator $\mathcal{Q}_2 = -sD_y + iy$ (we are denoting by $\mathcal{Q}_i$ the differential operators associated to the symmetries Y_i given in (11-12)), the following sequence of solutions to (8)

$$v_0 = 1\,, \quad v_1 = iy\,, \quad v_2 = -is - y^2\,, \quad v_3 = 3ys - iy^3\,, \quad \ldots \tag{14}$$

are obtained, according to Proposition 2. Notice that the operators $\mathcal{Q}_4$ and $\mathcal{Q}_s$ satisfy

$$\mathcal{Q}_4 = -i(\mathcal{Q}_2)^2 \quad , \quad \mathcal{Q}_s = -\frac{i}{2}(\mathcal{Q}_1)^2\,;$$

this is shown observing in particular that $D_{yy} = -2iD_s$ along all solutions to (8), therefore $\mathcal{Q}_4(v_0) = v_2$, *etc.*, whereas $\mathcal{Q}_1 = -D_y$ satisfies

$$\mathcal{Q}_1(v_n) = \lambda_n v_{n-1}\,, \quad \mathcal{Q}_1(v_0) = 0\,.$$

Similarly, one has, e.g., $\mathcal{Q}_3(v_n) = \mu_n v_n$ (where μ_n, λ_n are suitable constants), and $\mathcal{Q}_1\mathcal{Q}_2 = i(\mathcal{Q}_3 - 1)$, $[\mathcal{Q}_1, \mathcal{Q}_2] = -i$.

Other solutions to (8) could be obtained in this way, and/or applying Propositions 1 and 2, of course, but they are less relevant for our present purposes (see also below).

Alternatively, we can also look for the orbit of the solution $v_0 = 1$ under the (finite) action of the symmetry generator Y_2: it is obtained integrating the Lie equation (denoting by ε the real Lie parameter)

$$\frac{\mathrm{d}y}{\mathrm{d}\varepsilon} = s \qquad \frac{\mathrm{d}v}{\mathrm{d}\varepsilon} = iyv$$

with the "initial" condition (*i.e.*, for $\varepsilon = 0$) $v = v_0 = 1$; this gives

$$v(y, s, \varepsilon) = v_0 \exp(i\varepsilon y - \frac{i}{2}\varepsilon^2 s) = 1 + \varepsilon\, iy + \frac{\varepsilon^2}{2}(-is - y^2) + \frac{\varepsilon^3}{6}(3ys - iy^3) + \ldots$$

The coefficients of the powers ε^n are just the solutions obtained before.

The main interest in the symmetries and the particular solutions (14) found above for the free Schrödinger equation stems from the fact that they can be easily transformed into solutions of the Schrödinger equation for the harmonic oscillator (9). Indeed, using the transformation (10), one immediately obtains that the solutions (14) to the free equation become

$$\begin{gathered} u_0 = \exp(-x^2/2)\exp(-it/2)\,, \quad u_1 = \exp(-x^2/2)\exp(-3it/2)x\,, \quad \ldots \quad , \\ u_n = \exp(-x^2/2)\exp\big(-(n+1/2)t\big)H_n(x)\,, \quad \ldots \end{gathered} \tag{15}$$

where $H_n(x)$ are the $n-$degree Hermite polynomials, which are indeed solutions to the Schrödinger equation for the harmonic oscillator. More specifically, these are the solutions describing the stationary states of the quantum harmonic oscillator, which are eigenfunctions of the energy operator $i\partial/\partial t$ with eigenvalues (*i.e.*, with energy) $E_n = n + 1/2,\ n = 0, 1, 2, \ldots$. Let us recall that all normalizable $\big($*i.e.*, $\in L^2(\mathbf{R})\big)$ solutions of this equation can be expressed as convergent series $\big($in the $L^2(\mathbf{R})$ norm$\big)$ of the above set of solutions (15).

In the same way, one could directly obtain from the symmetry operators Y_i (11) and (12) for the free equation (8) the expressions of the Lie point symmetries X_i for the harmonic oscillator; however, using the transformation (10) which maps real into complex variables, we obtain the following "hybrid" expression for the generators where real and complex quantities are involved $\big($see the remark following eqs. (11,12); factors i, $i/2$ are put here just in order to obtain more convenient expressions for the generators $X_i\big)$

$$\begin{gathered} Y_v \to X_u = u\frac{\partial}{\partial u}\,, \quad Y'_v \to X'_u = i\,u\frac{\partial}{\partial u}\,, \quad Y_1 \to X_1 = \exp(it)\Big(\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}\Big)\,, \\ Y_2 \to \frac{i}{2}X_2 = \frac{i}{2}\exp(-it)\Big(\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}\Big)\,, \quad Y_3 \to i\,X_t - \frac{1}{2}X'_u = i\frac{\partial}{\partial t} - \frac{1}{2}\Big(i\,u\frac{\partial}{\partial u}\Big)\,, \\ Y_s \to i\,X_3 = i\,\exp(2it)\Big(x\frac{\partial}{\partial x} - i\frac{\partial}{\partial t} - \big(x^2 + \frac{1}{2}\big)u\frac{\partial}{\partial u}\Big)\,, \\ Y_4 \to \frac{i}{2}X_4 = \frac{i}{2}\exp(-2it)\Big(x\frac{\partial}{\partial x} + i\frac{\partial}{\partial t} + \big(x^2 - \frac{1}{2}\big)u\frac{\partial}{\partial u}\Big)\,. \end{gathered} \tag{16}$$

It is easy to transform the above operators $X_1, \ldots, X_4$ in the known form (see e.g. [15,20]) which preserves the property of x and t of remaining real variables under the transformations:

$$\widetilde{X}_1 = \cos t\frac{\partial}{\partial x} - x\sin t\Big(i\,u\frac{\partial}{\partial u}\Big)\,, \quad \widetilde{X}_2 = \sin t\frac{\partial}{\partial x} + x\cos t\Big(i\,u\frac{\partial}{\partial u}\Big)\,,$$
$$\widetilde{X}_3 = \cos(2t)x\frac{\partial}{\partial x} - \sin(2t)\frac{\partial}{\partial t} + x^2\sin(2t)\Big(i\,u\frac{\partial}{\partial u}\Big) + \frac{1}{2}\cos(2t)\Big(u\frac{\partial}{\partial u}\Big)\,,$$
$$\widetilde{X}_4 = \sin(2t)x\frac{\partial}{\partial x} - \cos(2t)\frac{\partial}{\partial t} + x^2\cos(2t)\Big(i\,u\frac{\partial}{\partial u}\Big) - \frac{1}{2}\sin(2t)\Big(u\frac{\partial}{\partial u}\Big)\,.$$

For completeness and in analogy with the nonlinear free equation (13), let us also consider the nonlinear version of the Schrödinger equation in the presence of the potential $V = x^2/2$:

$$i\,u_t + \frac{1}{2}u_{xx} - \frac{1}{2}x^2u + |u|^2u = 0\,. \tag{17}$$

It is easy to see that the surviving symmetries are the above X'_u, X_t, $\widetilde{X}_1$ and $\widetilde{X}_2$.

Let us now introduce, as before, the linear operators $\mathcal{R}_i$ corresponding to the generators in the form (16) (we adopt here the notation $\mathcal{R}$ instead of $\mathcal{Q}$ to avoid confusion). It is easily seen that, e.g. (using $2iD_t = -D_{xx} + x^2$ along the solutions)

$$\mathcal{R}_4 = \frac{1}{2}(\mathcal{R}_2)^2 \qquad\qquad \mathcal{R}_3 = -\frac{1}{2}(\mathcal{R}_1)^2$$

where $\mathcal{R}_2 = \exp(-i\,t)(-D_x + x)$, *etc.* As expected, the solution $u_0(x,t)$ is invariant under X_1 (*i.e.*, $\mathcal{R}_1(u_0) = 0$) and, as already remarked, corresponds, via the transformation (10), to the solution $v_0 = 1$ of the free eq. (8). Similarly, applying repeatedly $\mathcal{R}_2$ to u_0 one obtains the family of solutions $u_n(x,t)$ given in (15), and

$$\mathcal{R}_1(u_n) = \lambda_n\,u_{n-1}\,, \quad \mathcal{R}_2(u_n) = \nu_n\,u_{n+1} \tag{18}$$

where $\lambda_n\,,\nu_n$ are constants.

Then, the recursion operators defined by

$$\mathcal{R}_\downarrow := \mathcal{R}_1 = \exp(it)(-D_x - x) \quad , \quad \mathcal{R}_\uparrow := \mathcal{R}_2 = \exp(-it)(-D_x + x)$$

are exactly the Dirac step up/down operators well known in Quantum Mechanics [21]. These operators indeed satisfy the assumption of Proposition 3, with $A = i\partial/\partial t$. Their peculiar property, as expressed by eq. (18), is that of producing transitions from solutions u_n with energy E_n to the solutions $u_{n\pm1}$ with energy $E_{n\pm1} = E_n \pm 1$.

Although the application of the recursion operators $\mathcal{Q}_i$, associated to the symmetries Y_i (11-12) of the free equation, produces solutions of higher and higher degree, as we have seen, we must emphasize that these operators do *not* share with the operators $\mathcal{R}_\downarrow\,,\mathcal{R}_\uparrow$ the property of being ladder step up/down operators, *i.e.* they do not produce transitions between solutions with different energy levels. This depends not only on the fact that the energy spectrum for the free equation is a continuous one, as already remarked, but also that the transformation (10) does not map one into the other the energy operators $Y_s = \partial/\partial s$ and $X_t = \partial/\partial t$, as shown in (16), and this prevents the application of Proposition 3 to the case of the free equation.

3.2. The Schrödinger equation with $V(x) \propto x^2 + \delta/x^2$

Let us now examine the case in which the potential $V(x)$ in the Schrödinger equation has the form as in eq. (7); now, only two nontrivial independent symmetries are expected, according to Theorem 4.

By means of exactly the same transformation (10) used for the case $\delta = 0$, the Schrödinger equation (as before, putting $\beta = \gamma = x_0 = 0$ is not restrictive)

$$i\,u_t \;=\; -\frac{1}{2}u_{xx} + \frac{1}{2}x^2 u + \frac{\delta}{x^2}u \tag{19}$$

can be transformed into an equation where the term with x^2 disappears:

$$i\,v_s \;=\; -\frac{1}{2}v_{yy} + \frac{\delta}{y^2}v\;.$$

One could preliminarily give, proceeding as in Sect. 3.1, the symmetries and the solutions of this equation, but it is now simpler and actually much more interesting (thanks also to the final remark of the above subsection) to study directly eq. (19).

The two nontrivial independent symmetries of this equation are $\big($in the "hybrid" form as in (16); we adopt for this case the notation $X_a\,, X_b$, and $\mathcal{R}_a\,,\mathcal{R}_b$ for the associated operators$\big)$

$$X_a = \exp(2it)\Big(x\frac{\partial}{\partial x} - i\frac{\partial}{\partial t} - \big(x^2+\frac{1}{2}\big)u\frac{\partial}{\partial u}\Big)\,,\quad X_b \;=\; \exp(-2it)\Big(x\frac{\partial}{\partial x} + i\frac{\partial}{\partial t} + \big(x^2-\frac{1}{2}\big)u\frac{\partial}{\partial u}\Big) \tag{20}$$

which coincide with the symmetries $X_3\,,X_4$ found above for the case $\delta = 0$.

We look for the solution $u_0(x,t)$ which is invariant under X_a: substituting the condition $\mathcal{R}_a(u_0) = 0$ into eq. (19) one obtains an ordinary differential equation, which is easily solved to get

$$u_0 \;=\; x^\alpha \exp(-x^2/2)\exp\big(-it(\alpha+1/2)\big) \qquad \text{where} \quad \alpha = (1+\sqrt{1+8\delta})/2\;.$$

Applying then repeatedly to this solution the operator $\mathcal{R}_b$ one obtains the family of solutions

$$u_0(x,t),\quad u_1(x,t) \;=\; \mathcal{R}_b(u_0) \;=\; x^\alpha\exp(-x^2/2)(1+2\alpha-2x^2)\exp\big(-it(\alpha+5/2)\big),$$

$$u_2(x,t) \;=\; x^\alpha\exp(-x^2/2)(4x^4-12x^2-8\alpha x^2+4\alpha^2+8\alpha+3)\exp\big(-it(\alpha+9/2)\big)$$

and in general, for $n = 0,1,2,\ldots$,

$$u_n(x,t) \;=\; (\mathcal{R}_b)^n(u_0) \;=\; x^\alpha\exp(-x^2/2)P_n(x)\exp\big(-it(\alpha+1/2+2n)\big) \tag{21}$$

where $P_n(x)$ is a $2n$-degree polynomial, with energy eigenvalues

$$E_0 = \alpha + \frac{1}{2}\,,\quad E_1 = \alpha+\frac{5}{2}\,,\quad \ldots\,,\quad E_n = \alpha + 2n + \frac{1}{2}\,,\quad \ldots\;. \tag{22}$$

Notice that, also in this case and in agreement with Proposition 3, the recursion operators $\mathcal{R}_a\,,\mathcal{R}_b$ corresponding to $X_a\,,X_b$ are precisely step up/down operators which map any solution u_n to (19) having energy E_n to solutions $u_{n\pm1}$ with energy $E_{n\pm1} = E_n \pm 2$. $\big($Eq. (19), as far as eq. (9), admits of course other singular or divergent solutions, with no physical relevance and not included here.$\big)$

3.3. The effect of a centrifugal potential in $q > 1$ dimensions

The term $\propto x^{-2}$ in eq. (19) is reminiscent of the centrifugal potential: consider indeed the Schrödinger equation in $q > 1$ dimensions and assume that the potential depends only on $r = |\mathbf{x}|$, $\mathbf{x} \in \mathbf{R}^q$,

$$i\, u_t \;=\; -\frac{1}{2}\nabla^2 u + V(r)\, u\,;$$

the radial part of this equation is $\big($clearly, with a little abuse of notations, here $u = u(r,t)\big)$

$$i\, u_t \;=\; -\frac{1}{2}\Big(u_{rr} + \frac{q-1}{r}u_r - \frac{\ell(\ell+q-2)}{r^2}u\Big) + V(r)\, u \tag{23}$$

where $\ell(\ell+q-2)$ is the eigenvalue of the angular part of the q-dimensional Laplacian operator ∇^2 and ℓ is a fixed integer number $\ell = 0, 1, 2, \ldots$, which is interpreted in Quantum Mechanics as the angular momentum. However, in the above equation, in addition to the term $\propto r^{-2}$, there appears also a term $\propto r^{-1}$; but this one can be dropped by means of the transformation

$$u(r) \to w(r) \;=\; r^{(1-q)/2}\, u(r)$$

which indeed does transform (23) into

$$i\, w_t \;=\; -\frac{1}{2}w_{rr} + \frac{\delta_{cf}}{r^2}w + V(r) w$$

where now

$$\delta_{cf} = \frac{\ell(\ell+q-2)}{2} - \frac{q-1}{4} + \frac{(q-1)^2}{8} \;=\; \frac{1}{8}\big((2\ell+q-2)^2 - 1\big)\,.$$

In the presence of a potential of the form $V(r) = r^2/2 + \delta/r^2$, as before, the Schrödinger equation becomes then

$$i\, w_t \;=\; -\frac{1}{2}w_{rr} + \frac{1}{2}r^2 w + \frac{\delta_{eff}}{r^2}w$$

where $\delta_{eff} = \delta_{cf} + \delta$ is the "effective" coefficient which includes both the centrifugal potential δ_{cf}/r^2 and the "external" one δ/r^2.

It is important to point out that, in the "purely centrifugal" case, *i.e.* $\delta = 0$, the above equation is the radial part of the Schrödinger equation describing the isotropic harmonic oscillator in q dimensions. The solutions of this equation are well known, and can be obtained of course from (21-22) observing that in this case

$$\alpha \;=\; \frac{1}{2}\big(1 + \sqrt{1 + 8\delta_{cf}}\big) \;=\; \ell + \frac{q}{2} - \frac{1}{2}\,;$$

the energy eigenvalues for the isotropic q-dimensional harmonic oscillator turn out to be

$$E_0 = \frac{q}{2}\,, \quad E_1 = \frac{q}{2} + 1\,, \quad E_2 = \frac{q}{2} + 2\,, \quad E_3 = \frac{q}{2} + 3\,, \quad \ldots$$

having taken into account that ℓ may assume the values $0, 1, 2, \ldots$. Then, $E_{n+1} = E_n + 1$, whereas the step up/down operators $\mathcal{R}_a\,, \mathcal{R}_b$ associated to $X_a\,, X_b$ (20) are "double-step" operators, in agreement with Proposition 3, *i.e.* they connect each solution u_n having energy E_n with the solutions $u_{n\pm2}$, and not with the "consecutive" solutions $u_{n\pm1}$. This may appear surprising, but this depends on this fact: the operators $\mathcal{R}_a\,, \mathcal{R}_b$ necessarily connect solutions with the same quantum numbers ℓ, but it is well known that for each fixed n (*i.e.*, for a fixed value of the energy E_n), the admitted quantum numbers ℓ are either all even or all odd numbers, and if for a given n the numbers ℓ are, e.g., even, then for $n \pm 1$ they are all odd, and so on.

4. The Fokker–Planck equation

Another very important parabolic equation which can be treated by means of the above methods is the linear Fokker–Planck equation (see e.g. [22,23]), which may be written in a quite general form as follows

$$u_t \;=\; A(x)u + B(x)u_x + u_{xx} \tag{24}$$

where $A(x)$ and $B(x)$ are some given regular functions. As well known, equations of this type can be transformed into the prototypical form (2) by means of the transformation

$$u(x,t) \;=\; \omega(x)\, v(x,t)$$

where ω must satisfy $B\,\omega + 2\,\omega_x = 0$, and it is easy to conclude that the function $W(x)$ in (2) is then given by

$$W(x) \;=\; A + \frac{B^2}{4} - \frac{B_x}{2}\,.$$

Therefore, Theorem 4 states that eq. (24) admits nontrivial symmetries if (and only if) $A(x)$ and $B(x)$ satisfy

$$A + \frac{B^2}{4} - \frac{B_x}{2} \;=\; \alpha x^2 + \beta x + \gamma + \frac{\delta}{(x+x_0)^2}$$

and there are four nontrivial linearly independent symmetries if $\delta = 0$, and two symmetries if $\delta \neq 0$.

In addition, if the above conditions are verified, the Fokker–Planck equation can be transformed, by means of a transformation similar to (10) into the heat equation $v_s = v_{yy}$ in the first case (see also [24]), or into the equation $v_s = v_{yy} + (\delta/y^2)\,v$ if $\delta \neq 0$. We have already remarked, however, that the neat characterization of the step up/down operators is lost under this transformation, and we limit here our study to the equation in its original form.

Before looking for the symmetries and the solutions in this case, it is important to point out that in the context of the Fokker–Planck equation the eigenvalues of the operator $\partial/\partial t$ are no longer interpreted as the energy of the corresponding eigenfunctions (as it was the case of the Schrödinger equation), but anyway we shall see that we can completely recover the idea and the procedure of the step up/down operators simply looking for the solutions which are obtained by means of the separation of the variables, *i.e.* for the solutions of the form $u = T(t)\chi(x) = \exp(\kappa t)\chi_\kappa(x)$.

Several choices for the function A and B are of course possible (cf. e.g. [25–27] and references therein). We consider just two significant cases, to illustrate our procedure.

4.1. *The presence of four linearly independent nontrivial symmetries*

Let us choose $A = -1\,, B = -x$. Then one finds that the resulting equation

$$u_t \;=\; -u - xu_x + u_{xx}$$

admits the following nontrivial independent symmetries:

$$X_1 \;=\; \exp(t)\frac{\partial}{\partial x}\,, \quad X_2 \;=\; \exp(-t)\Big(\frac{\partial}{\partial x} + u\,x\frac{\partial}{\partial u}\Big)\,,$$

$$X_3 \,=\, \exp(2t)\Big(x\frac{\partial}{\partial x}+\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\Big)\,, \quad X_4 \,=\, \exp(-2t)\Big(x\frac{\partial}{\partial x}-\frac{\partial}{\partial t}+u\,x^2\frac{\partial}{\partial u}\Big)\,.$$

The solution u_0, invariant under X_1, *i.e.* $\mathcal{R}_1(u_0)=0$, with by now usual notation, is $u_0=\exp(-t)$; applying then $\mathcal{R}_2=\exp(-t)(-D_x+x)$ we obtain the set of solutions (apart from inessential factors)

$$\exp(-t)\,,\quad x\exp(-2t)\,,\quad (1{-}x^2)\exp(-3t)\,,\quad (3x{-}x^3)\exp(-4t)\,,\quad (3{-}6x^2{+}x^4)\exp(-5t)\,,\quad \ldots\,.$$

Notice that the recursion operators $\mathcal{R}_1\,,\mathcal{R}_2$ play here exactly the role of step up/down operators, with the difference that here their effect is to increase/decrease the "time-decay rate" κ of the solutions.

4.2. The presence of two independent nontrivial symmetries

We now choose in the Fokker–Planck equation $A=-1\,,B=-2(x+\frac{1}{x})$, which gives an equation

$$u_t \,=\, u-2\Big(x+\frac{1}{x}\Big)u_x+u_{xx}$$

admitting the two independent symmetries

$$X_a \,=\, \exp(4t)\Big(2x\frac{\partial}{\partial x}+\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}\Big)\,, \quad X_b \,=\, \exp(-4t)\Big(2x\frac{\partial}{\partial x}-\frac{\partial}{\partial t}+u(3+4x^2)\frac{\partial}{\partial u}\Big)\,.$$

In this case, we find *two* solutions to this equation which are invariant under $\mathcal{R}_a$, namely $u_0=\exp(-t)$ and $u_1=x^3\exp(-7t)$. Applying then to these the recursion operator $\mathcal{R}_b$ one obtains a *double* family of solutions:

$$\exp(-t)\,,\quad (1+2x^2)\exp(-5t)\,,\quad (1+4x^2-4x^4)\exp(-9t)\,,\quad \ldots$$

and

$$x^3\exp(-7t)\,,\quad (2x^5-5x^3)\exp(-11t)\,,\quad \ldots\ .$$

As in the previous case, the recursion operators $\mathcal{R}_a\,,\mathcal{R}_b$ play the role of step up/down operators, but here they change by 4 units the decay rate of the solutions. Notice that these operators cannot produce "jumps" from any solution in the first family above to any solution of the other (*i.e.*, no change by 2 units is admitted): this is because the operators $\mathcal{R}_a\,,\mathcal{R}_b$ are unaltered under the change $x\to -x$, and therefore cannot change the parity of the solutions. A similar situation has been encountered in the case of the Schrödinger equation in the presence of a centrifugal potential (Sect. 3.3).

5. Concluding remarks

In this paper we have considered several cases of two prototypical linear evolution equations: we have shown that their general symmetry properties are quite similar, but not completely identical; in particular, the presence of the imaginary factor i in the Schrödinger equation requires a special care. We have considered in detail the symmetry properties and provided families of solutions of the Schrödinger equation for the free particle, for the harmonic oscillator, and in the presence of centrifugal-like potentials. A comparison with two nonlinear Schrödinger equations is also presented. Two significant and different examples of the Fokker–Planck equation have been considered as well.

In all the linear cases considered, the equations can be transformed by means of a point transformation into a simplified form. On the one hand, this simpler form is convenient for finding solutions and symmetry properties, on the other hand, however, we have seen that in some sense it hides the peculiar property of these equations of admitting a set of simple linear recursion operators, which are related to the notion of step up/down operators and which admit a specific interpretation in terms of physically relevant quantities.

Acknowledgements

I am grateful to the referees for their useful comments and valuable suggestions.

References

1. Ovsjannikov, L.V. *Group properties of differential equations*; Siberian Acad. of Sciences: Novosibirsk, 1962, and *Group analysis of differential equations*, Academic Press: New York, NY, USA, 1982.
2. Stephani, H. *Differential equations. Their solution using symmetries*; Cambridge University Press: Cambridge, UK, 1989.
3. Olver, P.J. *Application of Lie groups to differential equations*,2nd ed.; Springer: Berlin, Germany, 1993.
4. Gaeta, G. *Nonlinear symmetries and nonlinear equations*; Kluwer: Dordrecht, The Netherlands, 1994.
5. Ibragimov, N.H., Ed. *CRC Handbook of Lie group analysis of differential equations*; CRC Press: Boca Raton, FL, USA, 1994, Vol. 1-3.
6. Bluman, G.W.; Anco, S.C. *Symmetry and integration methods for differential equations*; Springer: New York, NY, USA, 2002.
7. Fuchssteiner, B. Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Anal.* **1979**, *3*, 849-862.
8. Ma, W.-X.; Zhou, Z-X. Coupled integrable systems associated with a polynomial spectral problem and their Virasoro symmetry algebras. *Progr. Theor. Phys.* **1996**, *96*, 449-457.
9. Ma, W.-X. Extension of hereditary symmetry operators. *J. Phys. A: Math. Gen.* **1998**, *31*, 7279-7289.
10. Nucci, M. C. Nonclassical symmetries as special solutions of heir-equations. *J. Math. Anal. Appl.* **2003**, *279*, 168-179.
11. Cicogna, G. Symmetry classification of quasi-linear PDE's containing arbitrary functions. *Nonlinear Dynamics* **2008**, *51*, 309-316.
12. Cicogna, G. Symmetry classification of differential equations and reduction techniques. In *Lie groups: new research*; Canterra, A.B., Ed.; Nova Science Publishers: New York, NY, USA, 2009; pp. 385-399.
13. Kingston, J. G.; Sophocleous, C. On form-preserving point transformations of partial differential equations. *J. Phys. A: Math. Gen.* **1998**, *31*, 1597-1619.
14. Niederer, U. The maximal kinematical invariance group for the free Schrödinger equation. *Helv. Phys. Acta*, **1972/73**, *45*, 802-810.

15. Niederer, U. The maximal kinematical invariance group of the harmonic oscillator. *Helv. Phys. Acta* **1973**, *46*, 191-200.
16. Niederer, U. The maximal kinematical invariance groups of Schrödinger equations with arbitrary potentials. *Helv. Phys. Acta* **1974**, *46*, 167-172.
17. Boyer, C.P. The maximal 'kinematical' invariance group for an arbitrary potential. *Helv. Phys. Acta* **1974**, *45*, 589-605.
18. Ma, W.-X.; Chen, M. Direct search for exact solutions to the nonlinear Schrödinger equation. *Appl. Math. Comp.* **2009**, *215*, 2835-2842.
19. Fushchich, W.I.; Nikitin, A.G. *Symmetries of equations of quantum mechanics*; Allerton Press: New York, NY, USA, 1994.
20. Nucci, M.C.; Leach, P.G.L. Gauge variant symmetries for the Schrödinger equation. *Nuovo Cim. B* **2008**, *123*, 85-93.
21. Dirac, P.A.M. *The principles of quantum mechanics*; Clarendon Press: Oxford, UK, 1958.
22. van Kampen, N.G. *Stochastic processes in physics and chemistry*; North-Holland: Amsterdam, The Netherlands, 1981.
23. Haken, H. *Advanced synergetics*; Springer: Berlin, Germany, 1983.
24. Bluman, G.W. On the transformation of diffusion processes into the Wiener process. *SIAM J. Appl. Math.* **1980**, *39*, 238-247.
25. Sastry, C.C.A.; Dunn, K.A.; Rao, D.R.K.S. Ovsiannikov's method and the construction of partially invariant solutions. *J. Math. Phys.* **1987**, *28*, 1473-1480.
26. Rudra, P. Symmetry classes of Fokker–Planck type equations. *J. Phys. A: Math. Gen.* **1990**, *23*, 1663-1670.
27. Cicogna, G.; Vitali, D. Classification of the extended symmetries of Fokker–Planck equations. *J. Phys. A: Math. Gen.* **1990**, *23*, L85-L88.