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Complexity Phenomena Induced by Novel Symmetry and Symmetry-Breakings with Antiscreening at Cosmological Scales—A Tutorial

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Abstract: Complexity phenomena in cosmological evolution due to the scale-running of the propagator coupling constant can yield new insights related to virtual particles and antiscreening effects with dark matter consequences. This idea was developed in accordance with the differential-integral functional formulation of the Wilsonian renormalization group based on the one-particle irreducible scale-dependent effective action for gravitational evolution. In this tutorial communication, we briefly describe the essence of the result with minimal mathematical details and then consider a few simple examples to provide a basic understanding of such an interesting and intriguing complexity process in terms of fractional calculus.

Keywords: complexity; symmetry; virtual particles; antiscreening; renormalization group; fractional calculus; Lifshitz ordering; dark matter

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1. Introduction

It has been shown that if there is any scale-running of the gravitational constant, cosmological observations may experience antiscreening distortions. The theory, complexity-induced Lifshitz-ordering with multifractal antiscreening/screening (CILOMAS) [1], is based on the renormalization-group transformation at large scales due to the coarse-graining of classical fluctuations, symmetry-breakings, and related crossover effects. Such complexity phenomena can yield apparent non-Newtonian effects with altered dark matter estimates. We provide a brief review of these ideas. Then, by using fractional calculus and differintegral operators, we present calculations of simple antiscreening examples for structures endowed with the type of novel properties suggested by the theory. These results and their generalizations should be useful for cosmologists in interpreting the mass contents and related potential structures of their observations. Several exercises are included for the interested reader.

2. CILOMAS

The basic idea of CILOMAS emerged from the explicit analysis of the one-particle irreducible differential functional formulation of the renormalization group (RG) [2–4] in terms of the "gravitational effective action" based on the Wilsonian viewpoint [5]. The RG generator was applied to the cosmological evolution during the process of the formation of structures, with the gravitational constant treated as a running parameter. Such processes are induced by stochastic fluctuations, symmetry breakings, and spatiotemporal multifractal intermittency in the classical regime (i.e., at spatial scales much larger than the Planck scale). Thus, there is no need to worry about the usual small scale closure difficulty as in quantum gravity, and the RG generator works appropriately in the range of interest for both exact and perturbative calculations.

Complexity phenomena [6] can set in from an interacting dynamical system composed of many individual elements. If the interactions are nonlinear and long-range, quite often large-scale structures are generated—each (well developed or partially formed) structure being composed of many individual elements behaving more or less coherently together. Such "coherent structures" can have varied sizes. Moreover, the behavior resulting from the diversified interactions of these structures can become very complicated. Such is the situation in cosmological evolution [7]. As the masses in the cosmos evolve gravitationally in time under the initial influence of small fluctuations, we expect structures to begin to form due to linear and nonlinear gravitational instabilities and accompanying stochastic fluctuations. These entities will then interact and produce further enhanced fluctuations as well as new smaller and larger structures, a phenomenon vividly demonstrated over and over again via large scale ab initio numerical simulations based on, e.g., the ΛCDM (Lambda Cold Dark Matter) model. These effects, being generally sporadic and localized, will overwhelm any probable quantum gravitational effects, if any, particularly at large scales.

CILOMAS demonstrated that, for cosmological evolution, coherent structures of novel symmetry properties may be generated at very large scales. Such coherent structures span the "crossover" domain within the various admissible "generalized Lifshitz fixed points" induced by the scale running of the gravitational constant. These fixed points mark the locations of phase transitions among the admissible ordered or disordered, commensurate or incommensurate, isotropic or anisotropic coherent states. The resulting non-Newtonian behavior of the coherent structures can generally produce interesting "apparent" antiscreening effects at cosmological scales with dark matter implications.

The above represents a brief narrative account of CILOMAS for readers who have some nodding acquaintance with the basic ideas of Wilsonian formulation of the renormalization group. For mathematical details, interested readers are invited to consult [1] and references contained therein.

We shall now follow this thread and study the non-Newtonian effects in simple model calculations using fractional calculus and then suggest how these ideas may be generalized to analyze actual observational data. Because of the phenomena of complexity, the non-Newtonian effects can vary from one cosmological body (e.g., a galaxy cluster or halo) to another—an important result that differentiates CILOMAS from most of the other models with non-Newtonian interpretations.

3. Non-Newtonian Coherent States and Crossovers Based on Fractional Calculus

In this section, we shall consider examples of novel symmetry states, crossovers, and symmetry-breakings in cosmological evolution as suggested by the predictions of CILOMAS. As the propagator coupling parameter (the gravitational constant) acquires scale-running characteristics in renormalization-group analysis when viewed at cosmological scales, the gravitational evolution and resulting coherent structures exhibit various manifestations of apparent non-Newtonian behavior.

Equations of state for such structures can be calculated analytically and numerically using the functional differential renormalization group as demonstrated by Nicoll, Chang, Vvedensky, and their colleagues [4,8] and the references therein. On the other hand, most of these states and crossovers (aside from the explicit critical states near the fixed points) may also be adequately described by mean field theories. To study structures with fractional propagator powers, one may employ analytical techniques from the point of view of fractal geometry, fractional calculus, or a combination of these two approaches. Since we are primarily interested in the structures in three-dimensional space, it is more convenient to describe them in terms of phenomena induced by operators with fractional differential and integral characteristics in three dimensions.

Though rarely invoked in physical sciences, fractional calculus has a long history in classical applied mathematics. The easiest entry point to fractional calculus is the popular Riemann–Liouville integral form [9,10] for real $q < 0$:

$$\frac{d^q f}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x [x-y]^{-q-1} f(y) dy \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, and a is the lower limit of integration. Equation (1) can be deduced by generalizing the classical Cauchy formula for multiple integrations of integer orders n by identifying $-q$ with n. The extension of (1) to fractional differentiation for real $q \geq 0$ is achieved by choosing an integer n such that $q - n < 0$ and by requiring the differintegral operator

$$\frac{d^q f}{[d(x-a)]^q} \equiv \frac{d^n}{dx^n} \frac{d^{q-n} f}{[d(x-a)]^{q-n}}. \tag{2}$$

We shall discuss the generalization of Equations (1) and (2) near the end of this communication. For the present, Equations (1) and (2) will suffice for our simple applications to be discussed immediately below.

As mentioned in [1], the most important CILOMAS states are those pertaining to the potential ordering and corresponding mass distributions due to the scale-running of the gravitational constant. Consider, for example, the classical Poisson equation for the potential function $\phi(\mathbf{r})$:

$$(G)^{-1} \nabla^2 \phi(\mathbf{r}) = 4\pi\rho(\mathbf{r}). \tag{3}$$

For a given mass distribution $\rho(\mathbf{r})$, Equation (3) may be integrated to yield

$$\phi(\mathbf{r}) = -\int_{V'} \{G\rho(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|\} dV'. \tag{4}$$

When the gravitational constant attains scale-running characteristics, we may visualize a general differintegral gradient operation affecting the mass distribution. This will effectively provide a scale-running of the propagator (i.e., the Laplacian operator) with additional fractional-power vectorial momentum dependence. Depending on the order of the gradient operator (which can have anisotropic characteristics), the density distribution, and the corresponding potential function will take on "apparent" additional spatial variations.

3.1. Spherical Symmetry

As an example, consider the situation of a point source m at the origin. The radial Poisson equation for spherical isotropy becomes

$$G^{-1} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \frac{m}{r^2} \delta(r). \tag{5}$$

Now consider an isotropic fractional differintegral gradient operation (for $0 \geq q \geq -1$) of the operator $(d/dr)(r^2 d/dr)$ in the radial direction. This effectively introduces an altered "apparent" radial mass density:

$$\rho_q(r) \equiv m/v_q(r; r_0) \frac{d^q[\delta(r)]}{[dr]^q} = \frac{m r_0^q}{4\pi} \cdot \frac{r^{-q-3}}{\Gamma(-q)} \text{ for } r > 0 \tag{6}$$

where $v_q(r; r_0) = 4\pi r^2 r_0^{-q}$ and r_0 is an arbitrary constant of radial distance.

Thus, under the imposed scale-running of the differintegral operation, the point mass now exhibits an "apparent" power law distribution in the radial direction. In a sense, regions away from the origin have experienced an antiscreening effect. Due to the spherical geometry and multiplication factor of $v_q(r; r_0)$, the additional apparent (virtual) mass associated with the antiscreening effect can be quite significant.

We may now evaluate the corresponding "apparent" modification of the potential function due to such an antiscreening effect. The result may be obtained by solving the radial Poisson equation:

$$G^{-1} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi_q}{dr} \right) = 4\pi\rho_q(r). \tag{7}$$

The result for $(0 > q > -1, r > 0)$ and in the absence of a point source at the origin is:

$$\phi_q = [mGr_0^q/(q(1+q)\Gamma(-q))]r^{-q-1}. \tag{8}$$

Interested readers may wish to verify the expressions given in Equations (6) and (8) as exercises.

Such simple results with the adjustable differintegral order (crossover) parameter q are useful in interpreting observations related to spherically symmetric galaxy clusters and dwarf galaxy halos, and in resolving the discrepancies among measured mass distributions based on X-ray, weak gravitational lensing data, and other optical observations, as well as the more serious differences of density profiles between observational data and results from large-scale numerical simulations.

In fact, we may perform an order of magnitude estimate of the ratio between the total mass (M_2) detectable by gravitational means only $(r_1 > r > r_2)$ and the total mass (M_1) detectable by non-gravitational means $(r < r_1)$ based on the density profile given by Equation (6). Integrating over the respective spherical regimes, we obtain $M_2/M_1 = (r_2/r_1)^{-q} - 1$. Thus, over a wide range of the parameters, r_2/r_1 and q, we may find values of the mass ratio that are comparable to observations. For example, for $r_2/r_1 = 25$ and $q = -0.5$, the mass ratio is 4. The mass ratio and other non-Newtonian effects may be estimated more accurately based on nonlinear differintegral calculations that include other physics and non-Newtonian hydrodynamics as described in Section 3.3.

3.2. Anisotropic Situations

We now switch to the antiscreening effects generated by anisotropic fractional differintegral gradients of the mass distribution. For a uniaxial differintegral gradient in the z-direction, $d^q\rho/[d(z+\infty)]^q$ with $0 > q > -1$, that breaks the spherical symmetry of a point mass $\rho = m\delta(\mathbf{r})$ at the origin, the calculation has been considered previously by the authors of [11,12]. The resulting qth order mass distribution is given by

$$\rho_q = ml^q\delta(x)\delta(y)z^{-q-1}/\Gamma(-q) \text{ for } z > 0 \tag{9}$$

and zero for $z < 0$, where l is an arbitrary constant of length in the z-direction, and the lower limit of the fractional gradient is set at $-\infty$. Thus, an antiscreening effect is introduced for $z > 0$ analogous to that for the spherically symmetric case in the radial direction.

The corresponding "apparent" potential ϕ_q can be obtained by solving the Poisson equation: $\nabla^2\phi_q = 4\pi G\rho_q$. Because the lower limit of the differintegral operator is set at $-\infty$, the operators ∇^2 and $d^q/[d(z+\infty)]^q$ commute. Therefore, the "apparent" potential may be found by evaluating $-Gml^qd^q[1/r]/[d(z+\infty)]^q$. The resulting expression is azimuthal symmetric with respect to z. In spherical coordinates, it depends on the radial distance r and the polar angle ϑ with $z = r\cos\vartheta$. Thus, we may show that $\phi_q = R_q(r)I_q(h)$ with $h = \cos\theta$, $R_q(r) = (-mGl^q/\Gamma(-q))r^{-q-1}$, and

$$I_q(h) = \sum_{n=0}^{\infty}(2n+1)P_n(h)/[n(n+1) - q(1+q)]. \tag{10}$$

where $P_n(h)$ is the Legendre polynomial. The apparent potential has a similar dependence on r as the spherically symmetric case. Its symmetry is broken by the polar angle modulator $I_q(h)$ (see [11,12]). Readers may wish to carry out the intermediate algebra as an exercise.

Such uniaxial symmetry can of course be broken further by an orthogonal differintegral gradient of order q', e.g., in the y-direction or in the general r-direction of order q'' normal to z. The orders q, q', and q'' can be different and can be functions of $\mathbf{r}$. More generally, any linear combinations of such fractional gradients may be employed to achieve various non-symmetrical antiscreening effects. In fact, each of such differintegral gradients may have nonuniform or arbitrary periodic variations as prefactors that are allowed by the symmetry breakings of CILOMAS. Further generalization to other values of order q or different insertion points of the fractional gradient to the propagator as well as other choices of the initial mass distributions, can lead to even more interesting symmetry breakings. Readers may consider carrying out one of such generalizations as an exercise.

3.3. Further Nonlinear Generalizations and Evolution to Complexity

As mentioned in the original CILOMAS paper, there are other admissible symmetry breakings beside those generated by the "apparent" mass distributions. These symmetry breakings are generally governed by other physics and the continuum equations (e.g., non-Newtonian hydrodynamics), which can be derived from the renormalized gravitational effective action. These equations are nonlinear and subject to various initial and/or boundary conditions. Generally, such admissible dynamic, static, or stationary CILOMAS states need to be obtained through numerical calculations or simulations. Numerical schemes for fractional calculus may be formulated from the more rigorous and general definition of differintegrals by replacing the integrals in Equations (1) and (2) with differences and sums on a classically defined function f as described by [13]:

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{[(x-a)/N]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x-j[(x-a)/N])\right\}. \tag{11}$$

In such formulations, the equations of state must be determined self-consistently, as demonstrated by [4,8] and references therein.

4. Summary and Brief Conclusions

A narrative summary of the essence of the theory CILOMAS is provided. The theory suggests that due to the scale-running of the gravitational constant induced by classical stochastic fluctuations through renormalization-group transformations, a kaleidoscopic complexity of myriads of non-Newtonian coherent structures may be generated, with varied antiscreening effects producing "apparent" observational dark matter effects at cosmological scales. Utilizing the method of fractional calculus, simple analytical descriptions of such "apparent" non-Newtonian structures are presented as illustrations of the interesting effects of CILOMAS. Generalizations of these results are also discussed. Such possible observational distortions due to the scale running of the propagator constant may be useful to experimental cosmologists for their data interpretations. Several suggested exercises are included within this elementary tutorial for the interested readers.

We end this brief tutorial by mentioning that one of the reviewers of this manuscript kindly pointed out that Kleidis and Spyrou [14] have previously suggested that dynamical equivalence between geodesic motions and hydrodynamic flows could also introduce virtue mass terms and the modification of the gravitational potential. Their idea was based on conformal transformations, whereas the CILOMAS theory is based on the scale-running of the propagator coupling constant resulting from the classical RG transformations.

Conflicts of Interest: The author declares no conflict of interest.

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