Article

Symmetric Identities for Fubini Polynomials

Taekyun Kim 1,2, Dae San Kim 3, Gwan-Woo Jang 2 and Jongkyum Kwon 4,*

1 Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China; tdkim@kw.ac.kr or kwangwoonmath@hanmail.net
2 Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea; gwjiang@kw.ac.kr
3 Department of Mathematics, Sogang University, Seoul 121-742, Korea; dskim@sogang.ac.kr
4 Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo 52828, Korea
* Correspondence: mathkjk26@gnu.ac.kr

Received: 20 April 2018; Accepted: 13 June 2018; Published: 14 June 2018

Abstract: We represent the generating function of $w$-torsion Fubini polynomials by means of a fermionic $p$-adic integral on $\mathbb{Z}_p$. Then we investigate a quotient of such $p$-adic integrals on $\mathbb{Z}_p$, representing generating functions of three $w$-torsion Fubini polynomials and derive some new symmetric identities for the $w$-torsion Fubini and two variable $w$-torsion Fubini polynomials.

Keywords: Fubini polynomials; $w$-torsion Fubini polynomials; fermionic $p$-adic integrals; symmetric identities

1. Introduction and Preliminaries

In recent years, various $p$-adic integrals on $\mathbb{Z}_p$ have been used in order to find many interesting symmetric identities related to some special polynomials and numbers. The relevant $p$-adic integrals are the Volkenborn, fermionic, $q$-Volkenborn, and $q$-fermionic integrals of which the last three were discovered by the first author T. Kim (see [1–3]). They have been used by a good number of researchers in various contexts and especially in unfolding new interesting symmetric identities. This verifies the usefulness of such $p$-adic integrals. Moreover, we can expect that people will find some further applications of these $p$-adic integrals in the years to come. The present paper is an effort in this direction. Assume that $p$ is any fixed odd prime number. Throughout our discussion, we will use the standard notations $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Assume that $f(x)$ is a continuous function on $\mathbb{Z}_p$. Then the fermionic $p$-adic integral of $f(x)$ on $\mathbb{Z}_p$ was introduced by Kim (see [2]) as

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (1)$$

where $\mu_{-1}(x + p^N\mathbb{Z}_p) = (-1)^x$.

We can easily deduce from (1) that (see [2,3])

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0). \quad (2)$$

By invoking (2), we easily get (see [2,4])

$$\int_{\mathbb{Z}_p} e^{(x+y)} d\mu_{-1}(y) = \frac{2}{e^t+1} e^t = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (3)$$

Symmetry 2018, 10, 219; doi:10.3390/sym10060219
where \( E_n(x) \) are the usual Euler polynomials.  

As is known, the two variable Fubini polynomials are defined by means of the following (see [5,6])

\[
\sum_{n=0}^{\infty} F_n(x, y) t^n n! = \frac{1}{1 - y(e^t - 1)} e^{xt}. \tag{4}
\]

When \( x = 0 \), \( F_n(y) = F_n(0, y) \), \( n \geq 0 \), are called Fubini polynomials. Further, if \( y = 1 \), then \( Ob_n = F_n(0, 1) \) are the ordered Bell numbers (also called Frobenius numbers). They first appeared in Cayley’s work on a combinatorial counting problem in 1859 and have many different combinatorial interpretations. For example, the ordered Bell numbers count the possible outcomes of a multi-candidate election. From (3) and (4), we note that

\[
F_n(x, -1/2) = E_n(x), \quad (n \geq 0).
\]

By (4), we easily get (see [6]),

\[
F_n(y) = \sum_{k=0}^{n} S_2(n, k) k! y^k, \quad (n \geq 0), \tag{5}
\]

where \( S_2(n, k) \) are the Stirling numbers of the second kind.

For \( w \in \mathbb{N} \), we define the two variable \( w \)-torsion Fubini polynomials given by

\[
\int_{\mathbb{Z}_p} \left( -1 \right)^x (y(e^t - 1))^x d\mu_{-1}(x) = \frac{2}{1 - y(e^t - 1)} = 2 \sum_{n=0}^{\infty} F_n(y) t^n n!, \tag{7}
\]

and

\[
e^{xt} \int_{\mathbb{Z}_p} (-1)^z (y(e^t - 1))^z d\mu_{-1}(z) = \frac{2}{1 - y(e^t - 1)} e^{xt} = 2 \sum_{n=0}^{\infty} F_n(x, y) t^n n!. \tag{8}
\]

2. Symmetric Identities for \( w \)-torsion Fubini and Two Variable \( w \)-torsion Fubini Polynomials

From (2), we note that

\[
\int_{\mathbb{Z}_p} (-1)^x (y(e^t - 1))^x d\mu_{-1}(x) = \frac{2}{1 - y(e^t - 1)} = 2 \sum_{n=0}^{\infty} F_n(y) t^n n!, \tag{7}
\]

and

\[
e^{xt} \int_{\mathbb{Z}_p} (-1)^z (y(e^t - 1))^z d\mu_{-1}(z) = \frac{2}{1 - y(e^t - 1)} e^{xt} = 2 \sum_{n=0}^{\infty} F_n(x, y) t^n n!. \tag{8}
\]
From (7) and (8), we note that
\[
\left(\sum_{l=0}^{\infty} \frac{x^l}{l!}\right) \left(\sum_{m=0}^{\infty} 2F_m(y) \frac{t^m}{m!}\right) = e^{xt} \int_{\mathbb{Z}_p} (-1)^{x}(y(\varepsilon^l - 1))^x d\mu_{-1}(z)
\]
\[
= \sum_{n=0}^{\infty} 2F_n(x, y) \frac{t^n}{n!}.
\]
(9)

Thus, by (9), we easily get
\[
\sum_{l=0}^{n} \binom{n}{l} x^l F_{n-l}(y) = F_n(x, y), \ (n \geq 0).
\]
(10)

Now, we observe that
\[
\frac{1 - y^k(e^l - 1)^k}{1 - y(e^l - 1)} = \sum_{i=0}^{k-1} y^i(e^l - 1)^i = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} y^j \varepsilon^l t^i
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} y^j \varepsilon^l t^i \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{k-1} y^i \Delta^0 t^i \right) \frac{t^n}{n!}.
\]
(11)

where \(\Delta f(x) = f(x + 1) - f(x)\).

For \(w \in \mathbb{N}\), the \(w\)-torsion Fubini polynomials are represented by means of the following fermionic \(p\)-adic integral on \(\mathbb{Z}_p\):
\[
\int_{\mathbb{Z}_p} (-y^w(e^l - 1)^w)^x d\mu_{-1}(x) = \frac{2}{1 - y^w(e^l - 1)^w} = \sum_{n=0}^{\infty} 2F_n(y) \frac{t^n}{n!},
\]
(12)

From (7) and (12), we have
\[
\frac{\int_{\mathbb{Z}_p} (-y(e^l - 1)^x d\mu_{-1}(x))}{\int_{\mathbb{Z}_p} (-y^{w_1}(e^l - 1)^{w_1})^x d\mu_{-1}(x)} = \frac{1 - y^{w_1}(e^l - 1)^{w_1}}{1 - y(e^l - 1)} = \sum_{i=0}^{w_1-1} y^i(e^l - 1)^i
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{w_1-1} y^i \Delta^0 t^i \right) \frac{t^n}{n!}, \ (w_1 \in \mathbb{N}).
\]
(13)

For \(w_1, w_2 \in \mathbb{N}\), we let
\[
I = \frac{\int_{\mathbb{Z}_p} (-y^{w_1}(e^l - 1)^{w_1})^x d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} (-y^{w_2}(e^l - 1)^{w_2})^x d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} (-y^{w_1+w_2}(e^l - 1)^{w_1+w_2})^x d\mu_{-1}(x)}.
\]
(14)

Here it is important to observe that (14) has the built-in symmetry. Namely, it is invariant under the interchange of \(w_1\) and \(w_2\).

Then, by (14), we get
\[
I = \left( \int_{\mathbb{Z}_p} (-y^{w_1}(e^l - 1)^{w_1})^x d\mu_{-1}(x) \right) \times \left( \frac{\int_{\mathbb{Z}_p} (-y^{w_2}(e^l - 1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1+w_2}(e^l - 1)^{w_1+w_2})^x d\mu_{-1}(x)} \right).
\]
(15)
Theorem 1. For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( n \geq 0 \), we have

\[
\sum_{k=0}^{n} \sum_{i=0}^{w_1-1} \binom{n}{k} F_{n-k,w_1}(y) y^{w_2 i} \Delta^{w_2 i} q^k = \sum_{k=0}^{n} \sum_{i=0}^{w_2-1} \binom{n}{k} F_{n-k,w_2}(y) y^{w_1 i} \Delta^{w_1 i} q^k.
\]
Remark 1. In particular, for \( w_1 = 1 \), we have

\[
F_n(y) = \sum_{k=0}^{n} \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n}{k} F_{n-k,w_2}(y) y^i \Delta^i 0^k. \quad (22)
\]

By expressing \( l \) in a different way, we have

\[
I = \left( \int_{Z_L} (y^{w_1} (e^l - 1)^{w_1})^x d\mu_{-1}(x) \right) \times \left( \frac{\int_{Z_L} (y^{w_1} (e^l - 1)^{w_1})^x d\mu_{-1}(x)}{\int_{Z_L} (y^{w_1} (e^l - 1)^{w_1})^x d\mu_{-1}(x)} \right)
\]

\[
= \left( \int_{Z_L} (y^{w_1} (e^l - 1)^{w_1})^x d\mu_{-1}(x) \right) \times \left( \frac{1 - y^{w_1} (e^l - 1)^{w_1}}{1 - y^{w_1} (e^l - 1)^{w_1}} \right)
\]

\[
= \left( \sum_{i=0}^{w_1-1} y^{w_1 i} (e^l - 1)^{w_1 i} \right) \times \left( \frac{1}{1 - y^{w_1} (e^l - 1)^{w_1}} \right)
\]

\[
= \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_1} \sum_{j=0}^{w_2} \left( \frac{w_1 i}{l} \right) y^{w_1 i} (e^{l (w_1 i - j)}) \left( -1 \right)^j F_{n,w_2} (w_1 i - l, y) \right) \frac{l^n}{n!}
\]

Interchanging the roles of \( w_1 \) and \( w_2 \), by (14), we get

\[
I = \left( \int_{Z_L} (y^{w_2} (e^l - 1)^{w_2})^x d\mu_{-1}(x) \right) \times \left( \frac{\int_{Z_L} (y^{w_2} (e^l - 1)^{w_2})^x d\mu_{-1}(x)}{\int_{Z_L} (y^{w_2} (e^l - 1)^{w_2})^x d\mu_{-1}(x)} \right)
\]

\[
= \left( \int_{Z_L} (y^{w_2} (e^l - 1)^{w_2})^x d\mu_{-1}(x) \right) \times \left( \frac{1 - y^{w_2} (e^l - 1)^{w_2}}{1 - y^{w_2} (e^l - 1)^{w_2}} \right)
\]

\[
= \left( \sum_{i=0}^{w_2-1} y^{w_2 i} (e^l - 1)^{w_2 i} \right) \times \left( \frac{1}{1 - y^{w_2} (e^l - 1)^{w_2}} \right)
\]

\[
= \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1} \sum_{j=0}^{w_1} \left( \frac{w_2 i}{l} \right) y^{w_2 i} (e^{l (w_2 i - j)}) \left( -1 \right)^j F_{n,w_1} (w_2 i - l, y) \right) \frac{l^n}{n!}
\]

Hence, by Equations (23) and (24), we obtain the following theorem.

**Theorem 2.** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1 \) (mod \( 2 \)), \( w_2 \equiv 1 \) (mod \( 2 \)), \( n \geq 0 \), we have

\[
\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2} \binom{w_1}{l} (w_2 i - l, y) \right) \frac{l^n}{n!}
\]

Remark 2. Especially, if we take \( w_1 = 1 \), then by Theorem 2, we get

\[
F_n(y) = \sum_{i=0}^{w_1-1} \sum_{l=0}^{i} (i^l (-1)^l F_{n,w_2} (i - l, y) \right). \quad (26)
\]
3. Conclusions

In this paper, we introduced $w$-torsion Fubini polynomials as a generalization of Fubini polynomials and expressed the generating function of $w$-torsion Fubini polynomials by means of a fermionic $p$-adic integral on $\mathbb{Z}_p$. Then we derived some new symmetric identities for the $w$-torsion Fubini and two variable $w$-torsion Fubini polynomials by investigating a quotient of such $p$-adic integrals on $\mathbb{Z}_p$, representing generating functions of three $w$-torsion Fubini polynomials. It seems that they are the first double symmetric identities on Fubini polynomials. As was done, for example in [4,20,21], we expect that this result can be extended to the case of triple symmetric identities. That is one of our next projects.

Author Contributions: T.K. and D.S.K. conceived the framework and structured the whole paper; T.K. wrote the paper; G.-W.J. and J.K. checked the results of the paper; D.S.K. and J.K. completed the revision of the article.

Conflicts of Interest: The authors declare no conflict of interest.

References


© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).