On Special Kinds of Involute and Evolute Curves in 4-Dimensional Minkowski Space

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Abstract: Recently, extensive research has been done on evolute curves in Minkowski space-time. However, the special characteristics of curves demand advanced level observations that are lacking in existing well-known literature. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space. We consider (1,3)-evolute curves with respect to the casual characteristics of the (1,3)-normal plane that are spanned by the principal normal and the second binormal of the vector fields and the (0,2)-evolute curve that is spanned by the tangent and first binormal of the given curve. We restrict our investigation of (1,3)-evolute curves to the (1,3)-normal plane in four-dimensional Minkowski space. This research contribution obtains a necessary and sufficient condition for the curve possessing the generalized evolute as well as the involute curve. Furthermore, the Cartan null curve is also discussed in detail.

Keywords: evolute; involute curves; mate curves; minkowski space

MSC: 53A04; 53A35

1. Introduction

In the theory of curves, one of the important and interesting problems is the characterization of regular curves, in particular, the involute–evolute of a given curve. Evolutes and involutes (also known as evolvents) were studied by C. Huygens [1]. According to D. Fuchs [2], an involute of a given curve is a curve to which all tangents of the given curve are normal. He also defined the equation for an enveloping curve of the family of normal planes for a space curve. Suleyman and Seyda [3] determined the concept of parallel curves, which means that if the evolute exists, then the evolute of the parallel arc will also exist and the involute will coincide with the evolute. Brewster and David [4] stated that a curve is composed of two arcs with a common evolute, and the common evolute of two arcs must be a curve with only one tangent in each direction. In general, the evolute of a regular curve has singularities, and these points correspond to vertices. Emin and Suha [5] determined that an evolute Frenet apparatus can be formed by an involute apparatus in four dimensional Euclidean space, so, in this way, another orthonormal of the same space can be obtained. Shyuichi Izumiya [6] defined evolutes as the loci of singularities of space-like parallels and geometric properties of non-singular space-like hyper surfaces corresponding to the singularities of space-like parallels or evolutes. Takami Sato [7] investigated the singularities and geometric properties of pseudo-spherical evolutes of curves on a space-like surface in three-dimensional Minkowski-space. Marcos Craizer [8] stated that the iteration of involutes generates a pair of sequences of curves with respect to the Minkowski metric and its dual.

According to Boaventura Nolasco and Rui Pacheco [9], correspondence between plane curves and null curves in Minkowski three-space exists. He also described the geometry of null curves in terms of
the curvature of the corresponding plane curves. M. Turgut and S. Yılmaz [10] obtained the Frenet apparatus of a given curve by defining the space-like involute–evolute curve couple in Minkowski space-time. Some researchers have investigated evolute curves and their characterization in Minkowski space [11–16] as well as in Euclidean space. Many researchers have dealt with evolute–involute curves, but no research has been carried out on the Cartan null curve. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space. We obtained necessary and sufficient conditions for the curve possessing a generalized evolute as well as an involute.

2. Preliminaries

Consider the Minkowski space-time, \((E_4^1, G)\), where \(E_4^1 = \{y = (y_1, y_2, y_3, y_4) | y_i \in R\}\) and \(G = -dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\). For any \(M = (m_1, m_2, m_3, m_4)\) and \(N = (n_1, n_2, n_3, n_4) \in T_y E\). We denote \(M \cdot N = G(M, N) = m_1n_1 + m_2n_2 + m_3n_3 + m_4n_4\). Let \(I\) be an open interval in \(R\) and \(a : I \rightarrow E_4^1\) be a regular curve in \(E_4^1\) that is parameterized by the arc length parameter, \(s\), and \(\{T, N, B_1, B_2\}\) is the moving Frenet frame along \(a\), consisting of the tangent vector, \(T\); the principal normal vector, \(N\); the first binormal vector, \(B_1\), and the second binormal vector, \(B_2\), respectively, so that \(T \wedge N \wedge B_1 \wedge B_2\) coincides with the standard orientation of \(E_4^1\). Then, \(T \cdot T = e_1, N \cdot N = e_2, B_1 \cdot B_1 = e_3, B_2 \cdot B_2 = e_4, e_i e_j e_k e_4 = -1, e_i \in \{1, -1\}, i \in \{1, 2, 3, 4\}\).

In particular, the following conditions hold: \(T \cdot N = T \cdot B_1 = T \cdot B_2 = 0 = N \cdot B_1 = N \cdot B_2 = B_1 \cdot B_2 = 0\).

In accordance with reference [17], the Frenet–Serret formula for \(a\) in \(E_4^1\) is given by

\[
\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix} =
\begin{bmatrix}
0 & \epsilon_2 k_1 & 0 & 0 \\
-\epsilon_1 k_1 & 0 & \epsilon_3 k_2 & 0 \\
0 & -\epsilon_2 k_2 & 0 & -\epsilon_1 \epsilon_2 \epsilon_3 k_3 \\
0 & 0 & -\epsilon_3 k_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}.
\] (1)

We introduce some methodologies in this paper. At any point of \(a\), the plane spanned by \(\{T, B_1\}\) is called the \((0,2)\)-tangent plane of \(a\). The plane spanned by \(\{N, B_2\}\) is called the \((1,3)\)-normal plane of \(a\).

Let \(a : I \rightarrow E_4^1\) and \(a^*: I \rightarrow E_4^1\) be two regular curves in \(E_4^1\), where \(s\) is the arc-length parameter of \(a\). Denote \(s^* = f(s)\) to be the arc-length parameters of \(a^*\). For any \(s \in I\), if the \((0,2)\)-tangent plane of \(a\) at \(a(s)\) coincides with the \((1,3)\)-normal plane of \(a^*\) at \(a^*(s)\), then \(a^*\) is called the \((0,2)\)-involute curve of \(a\) in \(E_4^1\) and \(a\) is called the \((1,3)\)-evolute curve of \(a^*\) in \(E_4^1\).

An arbitrary curve, \(a(s)\) in \(E_4^1\), can locally be space-like, time-like, or null (light-like) if all of its velocity vectors, \(a'(s)\), are respectively space-like, time-like, or null [18]. A null curve, \(a\), is parametrized by the pseudo-arc \(s\) if \(g(a''(s), a''(s)) = 1\) [19]. On the other hand, a nonnull curve, \(a\), is parametrized by the arc-length parameter, \(s\), if \(g(a'(s), a'(s)) = \pm 1\). In accordance with references [19,20], if \(a\) is null Cartan curve, the Cartan Frenet frame is given by

\[
\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & 0 & 0 \\
k_2 & 0 & -k_1 & 0 \\
0 & -k_2 & 0 & k_3 \\
-k_3 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix},
\] (2)

where \(k_1(s) = 0\) if \(a(s)\) is a null straight line or \(k_1(s) = 1\) in all other cases. In this case, the next conditions hold: \(T \cdot T = B_1 \cdot B_1 = 0, N \cdot N = B_2 \cdot B_2 = 1, T \cdot N = T \cdot B_2 = N \cdot B_1 = N \cdot B_2 = B_1 \cdot B_2 = 0, T \cdot B_1 = 1\).

3. The \((0,2)\)-Involute Curve of a Given Curve in \(E_4^1\)

In this section, we proceed to study the existence and expression of the \((0,2)\)-involute curve of a given curve in \(E_4^1\).
Theorem 1. Let \( \alpha : I \to E^3_1 \) be a regular curve parameterized by arc-length \( s \) so that \( k_1, k_2 \) and \( k_3 \) are not zero. If \( \alpha \) possesses the (0,2)-involute mate curve, \( \alpha^* (s) = \alpha (s) + (\phi_0 - s) T(s) + \varphi B_1 (s) \), with \( \varphi \neq 0 \), then \( k_1, k_2 \) and \( k_3 \) satisfy

\[
\frac{k_2}{k_1} = \tau, \quad \frac{k_3}{k_1} = t_1 (\tau + \epsilon_1 \epsilon_3 t_2), \quad \tau = \frac{\phi_0 - s + \varphi T^2 t_2}{\varphi (1 - \epsilon_1 \epsilon_3 t_1^2)},
\]

where \( \phi_0, \varphi, t_1 \) and \( t_2 \) are given constants. Moreover, the three curvatures of \( \alpha^* \) are given by

\[
k_1^* = -\frac{\epsilon_1 \epsilon_4 \epsilon_5 f T^2}{\varphi (\tau + \epsilon_1 \epsilon_3 t_2)}, \quad k_2^* = \frac{f (\epsilon_4 \epsilon_5 T^3 \tau - \epsilon_2 \epsilon_5 T^2 t_2 - \epsilon_1 \epsilon_4 \epsilon_5 t_2^2)}{\varphi t_1 (\tau + \epsilon_1 \epsilon_3 t_2)}, \quad k_3^* = -\frac{\epsilon_4 \epsilon_5 f}{\varphi t_1},
\]

where \( f \neq 0 \). The associated Frenet frame are given by

\[
T^* = f t_3 (t_1 N + B_2), \quad N^* = f (T + t_2 B_1), \quad B_1^* = g t_3 (-N + t_1 B_2), \quad B_2^* = f (-t_2 T + B_1).
\]

Proof. Let \( \alpha : I \to E^3_1 \) be a regular curve with arc-length parameter \( s \) so that \( k_1, k_2 \) and \( k_3 \) are not zero. Suppose that \( \alpha^* : I \to E^3_1 \) is the (0,2)-involute curve of \( \alpha \). \( \{ T^*, N^*, B_1^*, B_2^* \} \) is the Frenet frame along \( \alpha^* \) and \( k_1^*, k_2^* \) and \( k_3^* \) are the curvatures of \( \beta^* \). Then

span \{ \( T, B_1 \) \} = span \{ \( N^*, B_2^* \) \}, span \{ \( N, B_2 \) \} = span \{ \( T^*, B_1^* \) \}.

Moreover, \( \alpha^* \) can be expressed as

\[
\alpha^* (s) = \alpha (s) + \phi (s) T(s) + \varphi (s) B_1,
\]

where \( \phi (s) \) and \( \varphi (s) \) are \( C^\infty \) functions on \( I \).

By differentiating (3) with respect to \( s \) and using the Frenet formula (1), we get

\[
f' T^* = (1 + \phi') \alpha (s) T(s) + \phi' (s) B_1 + \epsilon_2 (\phi k_1 - \varphi k_2) N - \epsilon_1 \epsilon_2 \epsilon_3 \varphi k_3 B_2.
\]

Taking the inner product on both sides of (4) with \( T \) and \( B_1 \), respectively, we get \( 1 + \phi' = 0 \) and \( \varphi' = 0 \), which implies that \( \varphi \) is constant and \( \phi = \phi_0 - s \), where \( \phi_0 \) is the integration constant. So, (4) turns into

\[
f' T^* = \epsilon_2 (\phi k_1 - \varphi k_2) N - \epsilon_1 \epsilon_2 \epsilon_3 \varphi k_3 B_2.
\]

If we denote

\[
\mu = \frac{\epsilon_2 (\phi k_1 - \varphi k_2)}{f'}, \quad \nu = \frac{-\epsilon_1 \epsilon_2 \epsilon_3 \varphi k_3}{f'},
\]

then (5) turns into

\[
T^* = \mu N + \nu B_2, \quad \mu^2 + \nu^2 = 1.
\]

Case 1: \( \varphi \neq 0 \). In this case, \( \nu \neq 0 \). \( \frac{\mu}{\nu} = t_1 \) implies that \( \mu = t_1 \nu \) and

\[
e_2 (\phi k_1 - \varphi k_2) = -\epsilon_1 \epsilon_2 \epsilon_3 \varphi t_1 k_3, \quad f' = -\epsilon_1 \epsilon_2 \epsilon_3 \varphi \nu^{-1} k_3, \quad \nu^2 = \frac{1}{1 + t_1^2}.
\]

By differentiating (7) with respect to \( s \) and using the Frenet formula (1), we get

\[
e_2^* f' k_1^* N^* = \mu' N + \epsilon_1 \epsilon_2 \epsilon_3 \varphi t_1 k_3 + \nu' B_2 + \epsilon_3 (\mu k_2 - \nu k_3) B_1.
\]

By taking the inner product from both sides of (9) with \( N \) and \( B_2 \), respectively, we get \( \mu' = 0 \) and \( \nu' = 0 \), which implies that \( \mu \) and \( \nu \) are constants. So, (9) turns into

\[
e_2^* f' k_1^* N^* = -\epsilon_1 \epsilon_2 \epsilon_3 \varphi t_1 k_3 + \nu (\mu k_2 - \nu k_3) B_1.
\]
Denote
\[ f = -\frac{\epsilon_1 \nu t_1 k_1}{e_2 f' k_1'}, \quad g = \frac{\epsilon_3 \nu (t_1 k_2 - k_3)}{e_2 f' k_1'}, \tag{11} \]
then (10) turns into
\[ N^* = f T + g B_1, \quad f^2 + g^2 = 1. \tag{12} \]
\[ \frac{\xi}{\nu} = \tau_2 \] implies that \( g = \tau_2 f \) and
\[ t_1 \tau_2 k_1 = -\epsilon_1 \epsilon_3 (t_1 k_2 - k_3), f^2 = \frac{1}{1 + \tau_2^2}. \tag{13} \]
From Equations (8) and (13), we have
\[ \tau := \frac{k_2}{k_1} = \frac{\frac{\phi}{\nu} + t_1^2}{1 - \epsilon_1 \epsilon_3 t_1^2} \tag{14} \]
\[ \frac{\nu}{\xi} = \tau_3 \] implies that \( \nu = \tau_3 f \). From (11), we get
\[ f' k_1' = -\epsilon_1 \epsilon_2 t_1 t_3 k_1, f_3^2 = \frac{1 + \nu^2}{1 + \tau_1^2}. \tag{15} \]
By differentiating (12) with respect to \( s \) using the Frenet formula (1), we get
\[ -\epsilon_1^* f' k_1^* T^* + \epsilon_3^* f' k_3^* B_1^* = f' T + \epsilon_2^* (f k_1 - g k_2) N + g' B_1 - \epsilon_1 \epsilon_2 \epsilon_3 g k_3 B_2. \tag{16} \]
By taking inner product on both side of (16) by \( T \) and \( B_1 \) respectively, we get \( f' = 0 \) and \( g' = 0 \), which implies that \( f \) and \( g \) are constants. In this case, (16) turns into
\[ \epsilon_3^* f' k_3^* B_1^* = \epsilon_3^* f' k_3^* T^* + \epsilon_2^* (f k_1 - t_2 k_2) N - \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 t_3^2 B_2 + \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 t_3^2 B_2. \tag{17} \]
By substituting (7) and (15) into (17), we get
\[ f' k_3^* B_1^* = f k_1 (\epsilon_4 t_3^2 T - \epsilon_2 \epsilon_3^2 t_2^2 - \epsilon_1 \epsilon_4^2 t_3^2) (-N + t_1 B_2). \tag{18} \]
From (18), we may choose that
\[ B_1^* = -\epsilon_4 \nu N + \epsilon_2 \mu B_2, f' k_2^* = t_3^{-1} k_1 (\epsilon_4 t_3^2 T - \epsilon_2 \epsilon_3^2 t_2^2 - \epsilon_1 \epsilon_4^2 t_3^2). \tag{19} \]
By differentiating (19) about \( s \) and using the Frenet formula (1), we get
\[ -\epsilon_2^* f' k_2^* N^* - \epsilon_1^* \epsilon_2^* \epsilon_3^* f' k_3^* B_1^* = \epsilon_1 \epsilon_4 \nu k_1 T - (\epsilon_3 \epsilon_4 \nu k_2 + \epsilon_2 \epsilon_3 \mu k_3) B_1, \tag{20} \]
from which we obtain
\[ \epsilon_3^* f' k_3^* B_2^* = (\epsilon_2^* f' k_2^* + \epsilon_1 \epsilon_4 \nu k_1) T + (\epsilon_3^* g' k_2^* - \epsilon_3 \epsilon_4 \nu k_2 - \epsilon_2 \epsilon_3 \mu k_3) B_1 = -t_3^{-1} k_1 (\nu + \epsilon_1 \epsilon_3 t_2) (-g T + f B_1). \tag{21} \]
From (21), we may choose that
\[ B_2^* = -g T + f B_1, f' k_3^* = -\epsilon_4 \nu t_3^{-1} k_1 (\nu + \epsilon_1 \epsilon_3 t_2). \tag{22} \]
From Equations (14), (15), (18) and (22), we can easily acquire our theorem. \( \Box \)

Case 2: If \(\phi = 0\), we have the following theorem.
Theorem 2. Let \( \alpha : I \to E^4_1 \) be a regular curve with arc-length parameter \( s \) so that \( k_1, k_2 \) and \( k_3 \) are not zero. If \( \alpha \) possesses the \((0,2)\)-involute mate curve \( \alpha^* = \alpha(s) + (\phi_0 - s)T(s) \), then \( k_1 \) and \( k_2 \) satisfy
\[
gk_1 + f k_2 = 0, \tag{23} \]
where \( \phi_0, f, \) and \( g \) are given constants.

Moreover, the three curvatures of \( \alpha^* \) are given by
\[
k_1^* = \frac{1}{e_1 e_2 e_3^2 f(s - \phi_0)}, k_2^* = -\frac{e_4 e_2^* g k_3}{e_2(s - \phi_0) k_1}, k_3^* = \frac{e_1^* e_4 f k_3}{e_2(s - \phi_0) k_1}. \tag{24} \]
The associated Frenet frames are given by
\[
T^* = -N, N^* = fT + gB_1, B_1^* = -B_2, B_2^* = -gT + fB_1. \tag{25} \]

In this case, (4) turns into
\[
a^*(s) = a(s) + (\phi_0 - s)T(s). \tag{26} \]

By differentiating (24) with respect to \( s \) and using the Frenet Formula (1), we get
\[
f' T^* = e_2(\phi_0 - s)k_1 N, \tag{27} \]
from which we may assume that
\[
f' = e_2(s - \phi_0)k_1, T^* = -N. \tag{28} \]

By differentiating the second equation of (26) about \( s \) and using the Frenet Formula (1), we get
\[
e_2^2 f'^2 k_1^* N^* = e_1 k_1 T - e_3 k_2 B_1. \tag{29} \]

Suppose that
\[
N^* = fT + gB_1, f = \frac{e_1 k_1}{e_2^2 f k_1^*}, g = -\frac{e_3 k_2}{e_2^2 f k_1^*}, f^2 + g^2 = 1. \tag{30} \]

It follows that
\[
\frac{k_2}{k_1} = -\frac{g}{f} e_1 e_3. \tag{31} \]

By differentiating (27) about \( s \), we obtain that \( f \) and \( g \) are constants:
\[
e_2^3 f'^2 k_2^* B_1^* = e_1' f' k_1^* T - e_2(fk_1 - gk_2)N - e_1 e_2 e_3 g k_3 B_2 = -e_2 g(f k_1 + g k_2)N + e_4 g k_3 B_2 = e_4 g k_3 B_2. \tag{32} \]

Suppose that
\[
B_1^* = -B_2, f' k_2^* = -e_4 e_3^2 g k_3. \tag{33} \]

By differentiating (30) about \( s \), we obtain
\[
e_2^3 f'^2 k_3^2 B_1^* = e_2 f' k_2^* N^* + e_3 k_3 B_1 = -e_1' e_4 k_3 [f g T - (1 - g^2) B_1] = -e_1' e_4 f k_3 (g T - f B_1). \tag{34} \]

Suppose that
\[
T^* \wedge N^* \wedge B_1^* \wedge B_2^* = T \wedge N \wedge B_1 \wedge B_2, \tag{35} \]
then
\[
B_2^* = -g T + f B_1, f' k_3^* = e_1' e_4 f k_3. \tag{36} \]

From Equations (27), (30) and (33), we have achieved the desired theorem.
Remark 1. Theorems 1 and 2 are quite different.

4. The (1,3)-Evolute Curve of a Given Curve in $E^4_3$

In this section, we want to study the (1,3)-evolute curve of a given curve in $E^4_3$.

Theorem 3. Let $\alpha: I \to E^4_3$ be a regular curve with arc length parameter $s$ so that $k_1, k_2$ and $k_3$ are not zero. If $\alpha$ possesses the (1,3)-evolute mate curve, $\alpha^* = (s) + \frac{1}{\sqrt{\kappa_2(s)}} [iN(s) + jB_2(s)] - \frac{1}{\sqrt{\kappa_3(s)}} B_2(s)$, then $k_1, k_2$ and $k_3$ satisfy $\epsilon_1 k_1 + \epsilon_3 (j k_2 - j k_3) = 0$, where $i$ and $j$ are given constants. Three curvatures of $\alpha^*$ are given by

$$k_1^* = -\epsilon_1 \epsilon_2 \sqrt{2} (ik_1), k_2^* = \sqrt{2} \frac{\epsilon_4 \epsilon_2 k_3 / (2i) - \epsilon_1 \epsilon_2 j k_1}{\sqrt{2}}, k_3^* = -\sqrt{2} \kappa_3 / (2i f'), f' = (1/ik_1).$$

The associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = (T + B_1) / \sqrt{2}, B_1^* = -jN + iB_2, B_2^* = (-T + B_1) / \sqrt{2}.$$

Proof. Let $\alpha: I \to E^4_3$ be a regular curve with arc-length parameter $s$ so that $k_1, k_2$ and $k_3$ are not zero. Let $\alpha^*: I \to E^4_3$ be the (1,3)-evolute curve of $\alpha$. \{T*, N*, B_1^*, B_2^*\} is the Frenet frame along $\alpha^*$ and $k_1^*, k_2^*$ and $k_3^*$ are the curvatures of $\alpha^*$. Then,

$$\text{span}\{T, B_1\} = \text{span}\{N^*, B_2^*\}, \text{span}\{N, B_2\} = \text{span}\{T^*, B_1^*\}. \quad (34)$$

Moreover, $\alpha^*$ can be expressed as

$$\alpha^* = (s) + p(s) N(s) + q(s) B_2, \quad (35)$$

where $p(s)$ and $q(s)$ are $C^\infty$ functions on $I$.

Differentiating (35) with respect to $s$ using Frenet Formula (1), we get

$$T^* f' = (1 - p \epsilon_1 k_1) T + p' N + q' B_2 + \epsilon_3 (p k_2 - q k_3) B_1. \quad (36)$$

By taking the inner product from both sides of (36) with $T$ and $B_1$, respectively, we get

$$f T^* = p' N + q' B_2, p = \frac{1}{\epsilon_1 k_1}, q = \frac{\epsilon_1 k_2}{k_1 k_3}. \quad (37)$$

Denote

$$i = \frac{u'}{f'}, j = \frac{v'}{f'}, \quad (38)$$

then (37) turns into

$$T^* = iN + jB_2, i^2 + j^2 = 1. \quad (39)$$

By differentiating (39) with respect to $s$ and using the Frenet formula (1), we get

$$\epsilon_1^2 f' k_1^* N^* = itN - \epsilon_1 ik_1 T + j' B_2 + \epsilon_3 (ik_2 - j k_3) B_1. \quad (40)$$

By taking inner product on both sides of (40) with $N$ and $B_2$ respectively, we get $i' = 0$ and $j' = 0$, which implies that $i$ and $j$ are constants.

From (38), we obtain

$$p = if + p_0 = \frac{1}{\epsilon_1 k_1}, q = jf + q_0 = \frac{\epsilon_1 k_2}{k_1 k_3}. \quad (41)$$

Moreover, (40) turns into

$$\epsilon_1^2 f' k_1^* N^* = -\epsilon_1 ik_1 T + \epsilon_3 (ik_2 - j k_3) B_1. \quad (42)$$
Denote
\[ r = -\frac{\epsilon_1 k_1}{\epsilon_2 f' k_1}, t = \frac{\epsilon_3 (ik_2 - jk_3)}{\epsilon_2 f' k_1}. \] (43)
then (42) turns into
\[ N^* = rT + t B_1, f' k_1^* = -\epsilon_1 \epsilon_2^{-1} r k_1, r^2 + r^2 = 1. \] (44)
Moreover, we have
\[ -\epsilon_1 \epsilon_2^3 t k_1 + \epsilon_3 (r k_2 - r j k_3) = 0. \] (45)

**Case 1:** \( t \neq 0 \). By differentiating (44) about \( s \) and using the Frenet Formula (1), we get
\[ -\epsilon_1 f' k_1^* T^* + \epsilon_2 f' k_2^* B_1^* = \epsilon_2 (r k_1 - t k_2) N - \epsilon_1 \epsilon_2 \epsilon_3 t k_3 B_2 + r' T + t' B_1. \] (46)
By taking inner product on both sides of (46) with \( T \) and \( B_1 \) respectively, we get \( r' = 0 \) and \( t' = 0 \), which implies that \( r \) and \( t \) are constants. In this case, (46) turns into
\[ f' k_2^* B_1^* = \left(\frac{\epsilon_2 \epsilon_3^2}{r} - \epsilon_1 \epsilon_4 \xi^2 k_1 - \epsilon_2 \epsilon_3 t k_2\right) N - \left(\epsilon_4 \epsilon_3 t k_3 + \epsilon_1 \epsilon_4 \xi k_1\right) B_2. \] (47)
Denote
\[ \sigma = (f' k_2^*)^{-1} \left(\frac{\epsilon_2 \epsilon_3^2}{r} - \epsilon_1 \epsilon_4 \xi^2 k_1 - \epsilon_2 \epsilon_3 t k_2\right), \zeta = (f' k_2^*)^{-1} \left(\epsilon_4 \epsilon_3 t k_3 + \epsilon_1 \epsilon_4 \xi k_1\right), \]
then (47) turns into
\[ B_1^* = \sigma N + \zeta B_2, \sigma^2 + \zeta^2 = 1. \] (49)

Since \( T^* \perp B_1^* \), it follows from (40) and (50) that \( \frac{\zeta}{\sigma} = -\frac{1}{1} \), which implies that
\[ \epsilon_1 i k_1 + \epsilon_3 (i k_2 - j k_3) = 0. \] (50)
From (45) and (50), we can see that
\[ i k_2 - j k_3 = -\epsilon_1 \epsilon_3 i k_1, (\epsilon_1 r - \epsilon_1 \epsilon_4 t) i k_1 = 0. \] (51)
Since \( t \neq 0 \), it follows from (51) that \( t = r \). Hence, (49) turns into
\[ B_1^* = -j N + i B_2, f' k_2^* = \epsilon_1 \epsilon_4 \xi k_1 + \epsilon_4 \epsilon_3 t k_3. \] (52)
By differentiating (52) about \( s \) using (1), we get
\[ -\epsilon_2 \epsilon_4 f' k_2^* N^* + \epsilon_4 f' k_3^* B_2^* = \epsilon_1 j k_1 T - \epsilon_3 (j k_2 + i k_3) B_1, \] (53)
from which we obtain
\[ f' k_3^* B_2^* = \epsilon_2 \epsilon_4 f' k_2^* N^* + \epsilon_1 j k_1 T - \epsilon_3 (j k_2 + i k_3) B_1, -\epsilon_4 \epsilon_3 \epsilon_4 t^2 k_3 (-T + B_1). \] (54)
It follows from (54) that
\[ B_2^* = -t T + r B_1, f' k_3^* = -\epsilon_4 \epsilon_3 \epsilon_4 t^2 k_3. \] (55)
From (43), (52) and (55), we can easily acquire our desired theorem. □

**Case 2:** If \( t = 0 \), we have the following theorem.
Theorem 4. Let \( \alpha : I \to E^4_1 \) be a regular curve parameterized by arc-length \( s \) so that \( k_1, k_2 \) and \( k_3 \) are not zero. If \( \alpha \) possesses the (1,3)-evolute mate curve, \( \alpha^* (s) = \alpha(s) + \frac{1}{i k_1(s)}[i N(s) + j B_2(s)] \), then \( k_2 \) and \( k_3 \) satisfy \( ik_2 - jk_3 = 0 \), where \( i \) and \( j \) are given constants. Moreover, the three curvatures of \( \alpha^* \) are given by

\[
k^*_1 = - \epsilon_1 \epsilon_2 ik_1 / f', k^*_2 = - \epsilon_1 \epsilon_2 jk_1 / f', k^*_3 = - \epsilon_3 i^{-1}k_3 / f', f' = (1 / ik_1).
\]

The associated Frenet frames are given by

\[
T^* = iN + jB_2, N^* = T, B_1^* = - jN + iB_2, B_2^* = B_1.
\]

Proof. For this case, we may suppose that

\[
N^* = T, f'k_1^* = - \epsilon_1 \epsilon_2 ik_1, \epsilon_3 (ik_2 - jk_3) = 0.
\]

From (41) and the third equation of (58), we acquire

\[
p = i(f + f_0) = \frac{\epsilon_1}{k_1} q = j(f + f_0) = \frac{\epsilon_1 j}{k_1}.
\]

By differentiating (58) about \( s \) and using (1), we get

\[
- \epsilon_1^* f'k_1^* T^* + \epsilon_3 f' k^*_2 B_1^* = \epsilon_2 k_1 N.
\]

It follows that we may choose

\[
B_1^* = - jN + iB_2, f'k_2^* = - \epsilon_1 \epsilon_2 jk_1.
\]

By differentiating (61) about \( s \) using the Frenet Formula (1) and third equation of (58), we get

\[
B_2^* = B_1, f'k_3^* = - \epsilon_3 (jk_2 + ik_3) = \epsilon_3 i^{-1}k_3.
\]

From (58), (61) and (62), we can easily acquire our desired theorem. \( \square \)

Remark 2. Theorems 3 and 4 are quite different.

5. The (1,3)-Evolute Curve of a Cartan Null Curve in \( E^4_1 \)

In this section, we proceed to study the existence and expression of the (1,3)-evolute curve of a given Cartan null curve in \( E^4_1 \). At any point of \( \alpha \), the plane spanned by \( \{N, B_2\} \) is called the (1,3)-normal plane of \( \alpha \).

Let \( \alpha : I \to E^4_1 \) and \( \alpha^* : I \to E^4_1 \) be two regular curves in \( E^4_1 \), where \( s \) is the arc-length parameter of \( \alpha \). Denote \( s^* = f(s) \) to be the arc-length parameters of \( \alpha^* \). For any \( s \in I \), if the (0,2)-tangent plane of \( \alpha \) at \( \alpha(s) \) coincides with the (1,3)-normal plane at \( \alpha^*(s) \) of \( \alpha \), then \( \alpha^* \) is called the (0,2)-involute curve of \( \alpha \) in \( E^4_1 \) and \( \alpha \) is called the (1,3)-evolute curve of \( \alpha^* \) in \( E^4_1 \).

Theorem 5. Let \( \alpha : I \to E^4_1 \) be a null Cartan curve with arc length parameter \( s \) so that \( k_1 = 1 \), and \( k_2, k_3 \) are not zero, if \( \alpha \) possesses the (1,3)-evolute mate curve, \( \alpha^* (s) = \alpha(s) + \frac{1}{i k_1(s)}[i N(s) + j B_2(s)] - \frac{1}{k_1(s)}B_2(s) \), then \( k_1, k_2 \) and \( k_3 \) satisfy \( i + ik_2 - jk_3 = 0 \), where \( i \) and \( j \) are given constants. Three curvatures of \( \alpha^* \) are given by

\[
k^*_1 = - \frac{\sqrt{2} (i)}{f'}, k^*_2 = \frac{\sqrt{2} [k_3 / (2i) - j]}{f'}, k^*_3 = - \sqrt{2} k_3 / (2if'), f' = (1 / i).
\]
Moreover, the associated Frenet frames are given by

\[ T^* = iN + jB_2, \quad N^* = (T + B_1)/\sqrt{2}, \quad B_1^* = -jN + iB_2, \quad B_2^* = (-T + B_1)/\sqrt{2}. \] (64)

**Proof.** Let \( a : I \to E^3_1 \) be a Cartan null curve parameterized by the pseudo-arc parameter \( s \) with curvatures \( k_1 = 1, \) and \( k_2 \) and \( k_3 \) are not zero. Let \( a^*: I \to E^3_3 \) be the (1,3)-evolute curve of \( a. \) Denote \( \{T^*, N^*, B_1^*, B_2^*\} \) as the Frenet frame along \( a^* \) and \( k_1^*, k_2^*, k_3^* \) as the curvatures of \( a^*. \) Then

\[ \text{span}\{T, B_1\} = \text{span}\{N^*, B_1^*\}, \text{span}\{N, B_2\} = \text{span}\{T^*, B_1^*\}. \] (65)

Moreover, \( a^* \) can be expressed as

\[ a^*(s) = a(s) + p(s)N(s) + q(s)B_2, \] (66)

where \( p(s) \) and \( q(s) \) are \( C^\infty \) functions on \( I. \) By differentiating (66) with respect to \( s \) using the Frenet Formula (2), we get

\[ T^* f' = (1 + pk_2 - qk_3)T + p'N + q'B_2 - pB_1. \] (67)

By taking the inner product on both sides of (67) with \( T \) and \( B_1, \) respectively, we get

\[ f'T^* = p'N + q'B_2, \quad p = 1, q = \frac{k_2}{k_3}. \] (68)

Denote

\[ i = \frac{p'}{f'}, j = \frac{q'}{f'}, \] (69)

then (68) turns into

\[ T^* = iN + jB_2, i^2 + j^2 = 1. \] (70)

By differentiating (70) with respect to \( s \) and using the Frenet formula (2), we get

\[ f'k_1^*N^* = i'N - iB_1 + j'B_2 + (ik_2 - jk_3)T. \] (71)

By taking the inner product on both sides of (71) with \( N \) and \( B_2 \) respectively, we get \( i' = 0 \) and \( j' = 0 \) which implies that \( i \) and \( j \) are constants. From (69), we get

\[ p = if + p_0 = 1, q = jf + q_0 = \frac{k_2}{k_3}. \] (72)

Moreover, (71) turns into

\[ f'k_1^*N^* = -iB_1 + (ik_2 - jk_3)T. \] (73)

Denote

\[ r = -\frac{i}{f'k_1^*}, t = \frac{(ik_2 - jk_3)}{f'k_1^*}, \] (74)

then (73) turns into

\[ N^* = rB_1 + tT, f'k_1^* = -r^{-1}i, r^2 + t^2 = 1. \] (75)

Moreover,

\[ ti + rik_2 - rjk_3 = 0. \] (76)

**Case 1:** \( t \neq 0. \) By differentiating (75) about \( s \) and using the Frenet Formula (2), we get

\[ -f'k_1^*T^* + f'k_2^*B_1^* = (tk_1 - rk_2)N + rk_3B_2 + r'T + t'B_1. \] (77)
By taking the inner product from both sides of (77) with \( T \) and \( B_1 \) respectively, we get \( r' = 0 \) and \( t' = 0 \), which implies that \( r \) and \( t \) are constants. In this case, (77) turns into
\[
f'k_2^*B_1^* = \left( \frac{r^2 - t^2}{r} - tk_2 \right)N - (tk_3 + \frac{ij}{r})B_2.
\] (78)

Denote
\[
\sigma = (f'k_2^*)^{-1}\left( \frac{r^2 - t^2}{r} - tk_2 \right), \quad \varsigma = (f'k_2^*)^{-1}(tk_3 - \frac{ij}{r}),
\] (79)
then (78) turns into
\[
B_1^* = \sigma N + \varsigma B_2, \quad \sigma^2 + \varsigma^2 = 1.
\] (80)

Since \( T^* \perp B_1^* \), it follows from (70) and (80) that \( \frac{\sigma}{\varsigma} = -\frac{i}{j} \), which implies that
\[
i + ik_2 - jk_3 = 0.
\] (81)

From (76) and (81), we can see that
\[
ijk_2 - jk_3 = 0.
\] (82)

Since \( t \neq 0 \), it follows from (82) that \( t = r \).

Hence (80) turns into
\[
B_1^* = -j + iB_2, \quad f'k_2^* = -\frac{j}{r} + \frac{t}{i}k_3.
\] (83)

By differentiating (83) about \( s \) using (2), we get
\[
-f'k_2^*N^* + f'k_3^*B_2^* = jB_1 - (jk_2 + ik_3)T.
\] (84)

From which we have
\[
f'k_3^*B_2^* = f'k_2^*N^* + jB_1 - (jk_2 + ik_3)T, \quad -\frac{t^2}{i}k_3(-T + B_1).
\] (85)

It follows from (85) that
\[
B_2^* = -tT + rB_1, f'k_3^* = -\frac{t}{i}k_3.
\] (86)

From (74), (83) and (86), we easily acquire our desired theorem. \( \Box \)

**Case 2:** For \( t = 0 \), we have the following theorem.

**Theorem 6.** Let \( \alpha : I \rightarrow E_4^1 \) be a null Cartan curve with arc-length parameter \( s \) so that \( k_1 = 1, k_2 \) and \( k_3 \) are not zero. If \( \alpha \) possesses the (1,3)-evolute mate curve, \( \alpha^* (s) = \alpha(s) + \frac{1}{k_2(s)}(iN(s) + jB_2(s)) \), then \( k_2 \) and \( k_3 \) satisfy \( ik_2 - jk_3 = 0 \), where \( i \) and \( j \) are given constants. Moreover, the three curvatures of \( \alpha^* \) are given by
\[
k_1^* = -i/f', k_2^* = -j/f', k_3^* = i^{-1}k_3/f', f' = \left( \frac{1}{r} \right).
\] (87)

The associated Frenet frames are given by
\[
T^* = iN + jB_2, N^* = T, B_1^* = -jN + iB_2, B_2^* = B_1.
\] (88)

**Proof.** For this case, we may suppose that
\[
N^* = T, f'k_1^* = -i, ik_2 - jk_3 = 0.
\] (89)
Moreover, from (72) and the third equation of (89), we get
\[ p = i(f + f_0) = \frac{1}{k_1}q = j(f + f_0) = \frac{j}{i}. \] (90)

By differentiating (89) about \( s \) and using (2), we get
\[ f'k_1^*T^* + f'k_2^*B_1^* = k_1N. \] (91)

It follows that we can choose
\[ B_1^* = -jN + iB_2, f'k_2^* = -j. \] (92)

By differentiating (92) about \( s \) using the Frenet Formula (2) and third equation of (89), we get
\[ B_2^* = B_1, f'k_3^* = -(jk_2 + ik_3) = -i^{-1}k_3. \] (93)

From (89), (92) and (93), we can easily acquire our desired theorem. \( \Box \)

**Remark 3.** Theorems 5 and 6 are quite different.

**Condition 2:**

**Theorem 7.** Let \( \alpha : I \to E^4_1 \) be a null Cartan curve with arc length parameter \( s \) so that \( k_1 = 1 \), and \( k_2, k_3 \) are not zero if \( \alpha \) possesses the (1,3)-evolute mate curve, \( \alpha^*(s) = \alpha(s) + \frac{1}{ik_1}iN(s) + \frac{1}{k_1}B_2(s) \). Then, \( k_1, k_2, k_3 \) satisfy \( ik_1 + i - jk_3 = 0 \), where \( i \) and \( j \) are given constants. Three curvatures of \( \alpha^* \) are given, as follows:

\[
k_1^* = -\sqrt{2}(ik_1)/f', k_2^* = \sqrt[4]{2}[k_3/(2i) - jk_1]/f', k_3^* = -\sqrt{2}k_3/(2if'), f' = (1/ik_1).\] (94)

The associated Frenet Frame are given by

\[ T^* = iN + jB_2, N^* = (T + B_1)/\sqrt{2}, B_1^* = -jN + iB_2, B_2^* = (-T + B_1)/\sqrt{2}. \] (95)

**Proof.** Let \( \alpha : I \to E^4_1 \) be a Cartan null curve parametrized by pseudo-arc parameter \( s \) with curvatures \( k_2 = 1 \), and \( k_1 \) and \( k_3 \) are not zero. Let \( \alpha^* : I \to E^4_1 \) be the (1,3)-evolute curve of \( \alpha \). Denote \( \{T^*, N^*, B_1^*, B_2^*\} \) as the Frenet frame along \( \alpha^* \) and \( k_1^*, k_2^* \) and \( k_3^* \) as the curvatures of \( \alpha^* \). Then,

\[ \text{span}\{T, B_1\} = \text{span}\{N^*, B_2^*\}, \text{span}\{N, B_2\} = \text{span}\{T^*, B_1^*\}. \] (96)

Moreover, \( \alpha^* \) can be expressed as
\[ \alpha^*(s) = \alpha(s) + p(s)N(s) + q(s)B_2, \] (97)

where \( p(s) \) and \( q(s) \) are \( C^\infty \) functions on \( I \). By differentiating (97) with respect to \( s \) using the Frenet Formula (2), we get
\[ T^*f' = (1 + p - qk_3)T + p'N + q'B_2 - pk_1B_1. \] (98)

By taking the inner product from both sides of (98) with \( T \) and \( B_1 \) respectively, we get
\[ f'T^* = p'N + q'B_2, p = \frac{1}{k_1}, q = \frac{1}{k_1k_3}. \] (99)

Denote
\[ i = \frac{p'}{f'}, j = \frac{q'}{f'}. \] (100)
then (99) turns into

\[ T^* = iN + jB_2, i^2 + j^2 = 1. \]  

(101)

By differentiating (101) with respect to \( s \) and using the Frenet formula (2), we get

\[ f'k_1^2 N^* = iT - iN - jk_3 B_1 + f' B_2 + (i - jk_3) T. \]  

(102)

By taking the inner product from both sides of (102) with \( N \) and \( B_2 \), respectively, we get \( i' = 0 \) and \( j' = 0 \) which implies that \( i \) and \( j \) are constants. From (100), we get

\[ p = if + p_0 = \frac{1}{k_1}, q =jf + q_0 = \frac{1}{k_1 k_3}. \]  

(103)

Moreover, (102) turns into

\[ f'k_1^2 N^* = -ik_1 B_1 + (i - jk_3) T. \]  

(104)

Denote

\[ r = \frac{-ik_1}{f'k_1^2}, t = \frac{(i - jk_3)}{f'k_1^2}, \]  

then (104) turns into

\[ N^* = rB_1 + tT, f'k_1^2 = -r^{-1}ik_1, r^2 + t^2 = 1. \]  

(106)

Moreover,

\[ tik_1 + rjk_3 = 0. \]  

(107)

Case 1: \( t \neq 0 \). By differentiating (106) about \( s \) and using the Frenet Formula (2), we get

\[ -f'k_1^2 T^* + f'k_2^2 B_1^* = (tk_1 - r) N + rk_3 B_2 + r' T + t'B_1. \]  

(108)

By taking the inner product on both sides of (108) with \( T \) and \( B_1 \), respectively, we get \( r' = 0 \) and \( t' = 0 \) which implies that \( r \) and \( t \) are constants. In this case, (108) turns into

\[ f'k_2^2 B_1^* = (\frac{r^2 - t^2}{r}) N - (tk_3 + \frac{ijk_1}{r}) B_2. \]  

(109)

Denote

\[ \sigma = (f'k_2^2)^{-1}(\frac{r^2 - t^2}{r} k_1 - t), \xi = (f'k_2^2)^{-1}(tk_3 - \frac{ijk_1}{r}), \]  

then (109) turns into

\[ B_1^* = \sigma N + \xi B_2, \sigma^2 + \xi^2 = 1. \]  

(111)

Since \( T^* \perp B_1^* \), it follows from (101) and (111) that \( \frac{\sigma}{\xi} = -\frac{t}{r} \), which implies that

\[ ik_1 + i - jk_3 = 0. \]  

(112)

From (107) and (112), we get

\[ i - jk_3 = -ik_1, (t - r)ik_1 = 0. \]  

(113)

Since \( t \neq 0 \), it follows from (113) that \( t = r \). Hence, (111) turns into

\[ B_1^* = -jN + iB_2, f'k_2^2 = \frac{j}{r} k_1 + \frac{r}{i} k_3. \]  

(114)

By differentiating (114) about \( s \) using (2), we get
\[-f'k_2^2 N^* + f'k_3^2 B_2^* = jk_1B_1 - (j + ik_3)T.\] (115)

From which we have
\[f'k_2^2 B_2^* = f'k_2^2 N^* + jk_1B_1 - (j + ik_3)T, -\frac{t^2}{i}k_3 (\mathcal{T} - B_1).\] (116)

It follows from (116) that
\[B_2^* = -\frac{t}{j}T + rB_1, f'k_3^2 = -\frac{t}{i} k_3.\] (117)

From (106), (114) and (117), we can easily acquire our desired theorem.

Case 2: For \(t = 0\), we have the following theorem.

**Theorem 8.** Let \( \alpha : I \to E_4^1 \) be a null Cartan curve with arc-length parameter \( s \) so that \( k_1 = 1 \), \( k_2 \) and \( k_3 \) are not zero. If \( \alpha \) possesses the \((1,3)\)-evolute mate curve \( \alpha^*(s) = \alpha(s) + \frac{1}{k_1(s)} [iN(s) + jB_2(s)] \), then \( k_2 \) and \( k_3 \) satisfy \( ik_2 - jk_3 = 0 \), where \( i \) and \( j \) are given constants. Moreover, the three curvatures of \( \alpha^* \) are given by
\[k^*_1 = -i / f', k^*_2 = -j / f', k^*_3 = i^{-1}k_3 / f', f' = \left(\frac{1}{\gamma}\right).\] (118)

The associated Frenet frames are given by
\[T^* = iN + jB_2, N^* = T, B_1^* = -jN + iB_2, B_2^* = B_1.\] (119)

**Proof.** In this case, we may suppose that
\[N^* = B_1, f'k_1^* = -ik_1, i - jk_3 = 0.\] (120)

Moreover, from (112) and the third equation of (120), we get
\[p = i(f + f_0) = \frac{1}{k_1}, q = j(f + f_0) = \frac{j}{ik_1}.\] (121)

By differentiating (120) about \( s \) and using (2), we get
\[f'k_1^* T^* + f'k_3 B_1^* = k_1 N.\] (122)

It follows that we can choose
\[B_1^* = -jN + iB_2, f'k_2^* = -jk_1.\] (123)

By differentiating (123) about \( s \) using the Frenet Formula (2) and using the third equation of (120), we get
\[B_2^* = B_1, f'k_3^* = -(j + ik_3) = j^{-1}.\] (124)

From (120), (123) and (124), we can easily acquire our desired theorem. \( \square \)

**Remark 4.** Theorems 7 and 8 are quite different.

6. Conclusions

This paper established new kinds of generalized evolute and involute curves in four-dimensional Minkowski space by providing the necessary and sufficient conditions for the curves possessing generalized evolute and involute curves. Furthermore, the study invoked a new type of \((1,3)\)-evolute and \((0,2)\)-evolute curve in four-dimensional Minkowski space. The study also provided a new kind of generalized null Cartan curve in four-dimensional Minkowski space. For this new type of curve, the study provided several theorems with necessary and sufficient conditions and obtained significant
results. The understanding of evolute curves with this new type evolute curve in four-dimensional Minkowski space will be beneficial for researchers in future studies. The designing of a framework for the involutes of order k of a null Cartan curve in Minkowski spaces will be considered in future work.

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**References**


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