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Symmetric Identities for \((P, Q)\)-Analogue of Tangent Zeta Function

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Abstract: The goal of this paper is to define the \((p, q)\)-analogue of tangent numbers and polynomials by generalizing the tangent numbers and polynomials and Carlitz-type \(q\)-tangent numbers and polynomials. We get some explicit formulas and properties in conjunction with \((p, q)\)-analogue of tangent numbers and polynomials. We give some new symmetric identities for \((p, q)\)-analogue of tangent polynomials by using \((p, q)\)-tangent zeta function. Finally, we investigate the distribution and symmetry of the zero of \((p, q)\)-analogue of tangent polynomials with numerical methods.

Keywords: tangent numbers; tangent polynomials; Carlitz-type \(q\)-tangent numbers; Carlitz-type \(q\)-tangent polynomials; \((p, q)\)-analogue of tangent numbers and polynomials; \((p, q)\)-analogue of tangent zeta function; symmetric identities; zeros

MSC: 11B68; 11S40; 11S80

1. Introduction

The field of the special polynomials such as tangent polynomials, Bernoulli polynomials, Euler polynomials, and Genocchi polynomials is an expanding area in mathematics (see [1–16]). Many generalizations of these polynomials have been studied (see [1,3–9,11–18]). Srivastava [14] developed some properties and \(q\)-extensions of the Euler polynomials, Bernoulli polynomials, and Genocchi polynomials. Choi, Anderson and Srivastava have discussed \(q\)-extension of the Riemann zeta function and related functions (see [5,17]). Dattoli, Migliorati and Srivastava derived a generalization of the classical polynomials (see [6]).

It is the purpose of this paper to introduce and investigate a new some generalizations of the Carlitz-type \(q\)-tangent numbers and polynomials, \(q\)-tangent zeta function, Hurwitz \(q\)-tangent zeta function. We call them Carlitz-type \((p, q)\)-tangent numbers and polynomials, \((p, q)\)-tangent zeta function, and Hurwitz \((p, q)\)-tangent zeta function. The structure of the paper is as follows: In Section 2 we define Carlitz-type \((p, q)\)-tangent numbers and polynomials and derive some of their properties involving elementary properties, distribution relation, property of complement, and so on. In Section 3, by using the Carlitz-type \((p, q)\)-tangent numbers and polynomials, \((p, q)\)-tangent zeta function and Hurwitz \((p, q)\)-tangent zeta function are defined. We also contains some connection formulae between the Carlitz-type \((p, q)\)-tangent numbers and polynomials and the \((p, q)\)-tangent zeta function. In Section 4 we give several symmetric identities about \((p, q)\)-tangent zeta function and Carlitz-type \((p, q)\)-tangent polynomials and numbers. In the following Section, we investigate the distribution and symmetry of the zero of Carlitz-type \((p, q)\)-tangent polynomials using a computer. Our paper ends with Section 6, where the conclusions and future developments of this work are presented. The following notations will be used throughout this paper.
• $\mathbb{N}$ denotes the set of natural numbers.
• $\mathbb{Z}_{0} = \{0, -1, -2, -3, \ldots \}$ denotes the set of nonpositive integers.
• $\mathbb{R}$ denotes the set of real numbers.
• $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical tangent numbers $T_{n}$ and tangent polynomials $T_{n}(x)$ are defined by the following generating functions (see [19])

$$\frac{2}{e^{2t} + 1} = \sum_{n=0}^{\infty} \frac{T_{n} t^{n}}{n!}, \quad (|2t| < \pi), \quad (1)$$

and

$$\left(\frac{2}{e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n}(x) \frac{t^{n}}{n!}, \quad (|2t| < \pi).$$

respectively. Some interesting properties of basic extensions and generalizations of the tangent numbers and polynomials have been worked out in [11,12,18–20]. The $(p,q)$-number is defined as

$$[n]_{p,q} = \frac{p^{n} - q^{n}}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^{2} + \cdots + p^{2}q^{n-3} + pq^{n-2} + q^{n-1}.$$  

It is clear that $(p,q)$-number contains symmetric property, and this number is $q$-number when $p = 1$. In particular, we can see $\lim_{q\to 1}|n|_{p,q} = n$ with $p = 1$. Since $[n]_{p,q} = p^{n-1}[n]_{q}$, we observe that $(p,q)$-numbers and $p$-numbers are different. In other words, by substituting $q$ by $\frac{q}{p}$ in the definition $q$-number, we cannot have $(p,q)$-number. Duran, Acikgoz and Araci [7] introduced the $(p,q)$-analogues of Euler polynomials, Bernoulli polynomials, and Genocchi polynomials. Araci, Duran, Acikgoz and Srivastava developed some properties and relations between the divided differences and $(p,q)$-derivative operator (see [1]). The $(p,q)$-analogues of tangent polynomials were described in [20]. By using $(p,q)$-number, we construct the Carlitz-type $(p,q)$-tangent polynomials and numbers, which generalized the previously known tangent polynomials and numbers, including the Carlitz-type $q$-tangent polynomials and numbers. We begin by recalling here the Carlitz-type $q$-tangent numbers and polynomials (see [18]).

**Definition 1.** For any complex $x$ we define the Carlitz-type $q$-tangent polynomials, $T_{n,q}(x)$, by the equation

$$F_{q}(t,x) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^{n}}{n!} = [2]_{q} \sum_{m=0}^{\infty} (-1)^{m} q^{m} e^{2m|x|}. \quad (3)$$

The numbers $T_{n,q}(0)$ are called the Carlitz-type $q$-tangent numbers and are denoted by $T_{n,q}$. Based on this idea, we generalize the Carlitz-type $q$-tangent number $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$. It follows that we define the following $(p,q)$-analogues of the Carlitz-type $q$-tangent number $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$. In the next section we define the $(p,q)$-analogue of tangent numbers and polynomials. After that we will obtain some their properties.

2. $(p,q)$-Analogue of Tangent Numbers and Polynomials

Firstly, we construct $(p,q)$-analogue of tangent numbers and polynomials and derive some of their relevant properties.

**Definition 2.** For $0 < q < p \leq 1$, the Carlitz-type $(p,q)$-tangent numbers $T_{n,p,q}$ and polynomials $T_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t) = \sum_{n=0}^{\infty} T_{n,p,q} \frac{t^{n}}{n!} = [2]_{p,q} \sum_{m=0}^{\infty} (-1)^{m} q^{m} e^{2m|p,q|} t^{m}, \quad (4)$$
Theorem 1. For $n \in \mathbb{N} \cup \{0\}$, one has

$$T_{n,p,q} = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}p^{2(n-l)}}.$$ (6)

Proof. By (4), we have

$$\sum_{n=0}^{\infty} T_{n,p,q} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]_{p,q} t},$$ (5)

respectively.

Setting $p = 1$ in (4) and (5), we can obtain the corresponding definitions for the Carlitz-type $q$-tangent numbers $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$ respectively. Obviously, if we put $p = 1$, then we have

$$T_{n,p,q}(x) = T_{n,q}(x), \quad T_{n,p,q} = T_{n,q}.$$ (7)

Putting $p = 1$, we have

$$\lim_{q \to 1} T_{n,p,q}(x) = T_n(x), \quad \lim_{q \to 1} T_{n,p,q} = T_n.$$ (8)

Theorem 2. For $n \in \mathbb{N} \cup \{0\}$, one has

$$T_{n,p,q}(x) = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}p^{2(n-l)}}.$$ (9)

Proof. By (5), we obtain

$$T_{n,p,q}(x) = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}p^{2(n-l)}}.$$ (10)
Again, by using (5) and (8), we obtain

\[
\sum_{n=0}^{\infty} T_n, p,q (x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ \frac{[2]_q}{p - q} \right] \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{x_l} \frac{1}{1 + q^{2l+1} p^{2(n-l)}} \frac{t^n}{n!} 
\]

(9)

\[
= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+n]p,q} t^i.
\]

Since \([x + 2y]_{p,q} = p^{2y}[x]_{p,q} + q^x[2y]_{p,q}\), we have

\[
T_n, p,q (x) = [2]_q \sum_{l=0}^{n} \binom{n}{l} x^{n-l} q^x \sum_{k=0}^{l} \binom{l}{k} (-1)^k \frac{1}{p - q} \frac{1}{1 + q^{2k+1} p^{2(n-k)}} \frac{t^n}{n!} \]

(10)

By using (9) and (10), \((p,q)\)-number, and the power series expansion of \(e^{q^x}\), we give Theorem 2.

Furthermore, by (7) and Theorem 2, we have

\[
T_n, p,q (x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} q^x \sum_{k=0}^{l} \binom{l}{k} (-1)^k \frac{1}{p - q} \frac{1}{1 + q^{2k+1} p^{2(n-k)}} \frac{t^n}{n!}.
\]

From (4) and (5), we can derive the following properties of the Carlitz-type tangent numbers \(T_n, p,q\) and polynomials \(T_n, p,q (x)\). So, we choose to omit the details involved.

**Proposition 1.** For any positive integer \(n\), one has

1. \(T_n, p,q (x) = \frac{[2]_q}{[2]_q} \frac{1}{q^m} \sum_{m=0}^{n} (-1)^m q^m T_m, p,q (n + \frac{2m + x}{m})\), \((m = \text{odd})\).

2. \(T_n, p^{-1}, q^{-1} (2 - x) = (-1)^n p^n q^n T_n, p,q (x)\).

**Theorem 3.** For \(n \in \mathbb{N} \cup \{0\}\), one has

\[
q T_n, p,q (2) + T_n, p,q = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
\]

**Theorem 4.** If \(n\) is a positive integer, then we have

\[
\sum_{l=0}^{n-1} (-1)^l q^l [2]_q^m T_n, p,q (2n) + T_m, p,q = \frac{(-1)^n q^n T_n, p,q (2n)}{[2]_q}.
\]

**Proof.** By (4) and (5), we get

\[
- [2]_q \sum_{l=0}^{\infty} (-1)^l q^l e^{[2l+2m]p,q} t^l = [2]_q \sum_{l=0}^{\infty} (-1)^l q^l e^{[2l+2m]p,q} t^l = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l e^{[2l+2m]p,q} t^l.
\]

(11)
Hence, by (4), (5) and (11), we have

\[
(-1)^{n+1}q^n \sum_{m=0}^{\infty} T_{m,p,q}(2n) \frac{i_m}{m!} + \sum_{m=0}^{\infty} T_{m,p,q} \frac{i_m}{m!}
\]

\[
= \sum_{m=0}^{\infty} \left( [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [2l]_{p,q} \right) \frac{i_m}{m!}.
\]

Equating coefficients of \( \frac{i_m}{m!} \) gives Theorem 4. \( \square \)

3. \((p, q)\)-Analogue of Tangent Zeta Function

Using Carlitz-type \((p, q)\)-tangent numbers and polynomials, we define the \((p, q)\)-tangent zeta function and Hurwitz \((p, q)\)-tangent zeta function. These functions have the values of the Carlitz-type \((p, q)\)-tangent numbers \( T_{n,p,q} \), and polynomials \( T_{n,p,q}(x) \) at negative integers, respectively. From (4), we note that

\[
\frac{d^k}{dt^k} T_{p,q}(t) \bigg|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^n q^m [2m]_{p,q}^k
\]

\[
= T_{k,p,q}, \quad (k \in \mathbb{N}).
\]

From the above equation, we construct new \((p, q)\)-tangent zeta function as follows:

**Definition 3.** We define the \((p, q)\)-tangent zeta function for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \) by

\[
\zeta_{p,q}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[2n]^2_{p,q}}.
\]

Notice that \( \zeta_{p,q}(s) \) is a meromorphic function on \( \mathbb{C} (\text{cf}.7) \). Remark that, if \( p = 1, q \to 1 \), then \( \zeta_{p,q}(s) = \zeta_T(s) \) which is the tangent zeta function (see [19]). The relationship between the \( \zeta_{p,q}(s) \) and the \( T_{k,p,q} \) is given explicitly by the following theorem.

**Theorem 5.** Let \( k \in \mathbb{N} \). We have

\[
\zeta_{p,q}(-k) = T_{k,p,q}.
\]

Please note that \( \zeta_{p,q}(s) \) function interpolates \( T_{k,p,q} \) numbers at non-negative integers. Similarly, by using Equation (5), we get

\[
\frac{d^k}{dt^k} T_{p,q}(t, x) \bigg|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m + x]_{p,q}^k \quad (13)
\]

and

\[
\left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,p,q}(x) \frac{t^n}{n!} \right) \bigg|_{t=0} = T_{k,p,q}(x), \quad \text{for } k \in \mathbb{N}. \quad (14)
\]

Furthermore, by (13) and (14), we are ready to construct the Hurwitz \((p, q)\)-tangent zeta function.

**Definition 4.** For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \) and \( x \notin \mathbb{Z}^- \), we define

\[
\zeta_{p,q}(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[2n + x]^2_{p,q}}.
\]

Obverse that the function \( \zeta_{p,q}(s, x) \) is a meromorphic function on \( \mathbb{C} \). We note that, if \( p = 1 \) and \( q \to 1 \), then \( \zeta_{p,q}(s, x) = \zeta_T(s, x) \) which is the Hurwitz tangent zeta function (see [19]). The function
\(\zeta_{p,q}(-k, x)\) interpolates the numbers \(T_{k,p,q}(x)\) at non-negative integers. Substituting \(s = -k\) with \(k \in \mathbb{N}\) into (15), and using Theorem 2, we easily arrive at the following theorem.

**Theorem 6.** Let \(k \in \mathbb{N}\). One has
\[
\zeta_{p,q}(-k, x) = T_{k,p,q}(x).
\]

4. Some Symmetric Properties About \((P, Q)\)-Analogue of Tangent Zeta Function

Our main objective in this section is to obtain some symmetric properties about \((p,q)\)-tangent zeta function. In particular, some of these symmetric identities are also related to the Carlitz-type \((p,q)\)-tangent polynomials and the alternate power sums. To end this section, we focus on some symmetric identities containing the Carlitz-type \((p,q)\)-tangent zeta function and the alternate power sums.

**Theorem 7.** Let \(w_1\) and \(w_2\) be positive odd integers. Then we have
\[
\begin{align*}
[2]_{q^w_1} [w_1]_{p,q}^s & \sum_{i=0}^{w-1} (-1)^i q^{w_1 i} \zeta_{p^2,q^2}^s \left( s, w_1 x + \frac{2w_1 i}{w_2} \right) \\
& = [2]_{q^w_2} [w_2]_{p,q}^s \sum_{i=0}^{w-1} (-1)^i q^{w_2 i} \zeta_{p^2,q^2}^s \left( s, w_2 x + \frac{2w_2 i}{w_1} \right).
\end{align*}
\]

**Proof.** For any \(x, y \in \mathbb{C}\), we observe that \([xy]_{p,q} = [x]_{p^2,q^2}[y]_{p,q}\). By substituting \(w_1 x + \frac{2w_1 i}{w_2}\) for \(x\) in Definition 4, replace \(p\) by \(p^2\) and replace \(q\) by \(q^2\), respectively, we derive
\[
\begin{align*}
\zeta_{p^2,q^2}^s \left( s, w_1 x + \frac{2w_1 i}{w_2} \right) &= [2]_{q^w_2} [w_2]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{w_1 x + \frac{2w_2 n}{w_2} + 2n}^s_{p^2,q^2} \\
& = [2]_{q^w_2} [w_2]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{w_1 w_2 x + 2w_1 i + 2w_2 n}^s_{p,q}.
\end{align*}
\]

Since for any non-negative integer \(m\) and positive odd integer \(w_1\), there exist unique non-negative integer \(r\) such that \(m = w_1 r + j\) with \(0 \leq j \leq w_1 - 1\). Thus, this can be written as
\[
\begin{align*}
\zeta_{p^2,q^2}^s \left( s, w_1 x + \frac{2w_1 i}{w_2} \right) &= [2]_{q^w_2} [w_2]_{p,q}^s \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{w_1 r+j} q^{w_2 (w_1 r+j)}}{w_1 w_2 (2w_1 r + x) + 2w_1 i + 2w_2 j}^s_{p,q} \\
& = [2]_{q^w_2} [w_2]_{p,q}^s \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{w_1 r+j} q^{w_2 (w_1 r+j)}}{w_1 w_2 (2r + x) + 2w_1 i + 2w_2 j}^s_{p,q}.
\end{align*}
\]

It follows from the above equation that
\[
\begin{align*}
[2]_{q^w_1} [w_1]_{p,q}^s & \sum_{i=0}^{w-1} (-1)^i q^{w_1 i} \zeta_{p^2,q^2}^s \left( s, w_1 x + \frac{2w_1 i}{w_2} \right) \\
& = [2]_{q^w_1} [2]_{q^w_2} [w_1]_{p,q}^s [w_2]_{p,q}^s \\
& \times \sum_{i=0}^{w-1} \sum_{j=0}^{w-1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{w_1 w_2 r + w_1 i + w_2 j}}{w_1 w_2 (2r + x) + 2w_1 i + 2w_2 j}^s_{p,q}.
\end{align*}
\]
From the similar method, we can have that

\[ \zeta_{p^m, q^n}(s, w_2 x + \frac{2w_2 j}{w_1}) = [2]_{q^n} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{w_2 x + \frac{2w_2 j}{w_1} + 2n} \]

\[ = [2]_{q^n} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{w_1 w_2 x + 2w_2 j + 2w_1 n} \]

Thus, from (16) and (17), we obtain the result. 

Thus, by (18) and (19), this concludes our proof.

**Corollary 1.** For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), we have

\[ \zeta_{p,q}(s, w_1 x) = [w_1]_{p,q} \sum_{j=0}^{w_1 - 1} (-1)^j q^j \zeta_{p^m, q^n}(s, x + \frac{2j}{w_1}) \]

**Proof.** Let \( w_2 = 1 \) in Theorem 7. Then we immediately get the result.

Next, we also derive some symmetric identities for Carlitz-type \((p, q)\)-tangent polynomials by using \((p, q)\)-tangent zeta function.

**Theorem 8.** Let \( w_1 \) and \( w_2 \) be any positive odd integers. The following multiplication formula holds true for the Carlitz-type \((p, q)\)-tangent polynomials:

\[ [2]_{q^n} \sum_{j=0}^{w_2 - 1} (-1)^j q^j T_{n, p^m, q^n}(w_1 x + \frac{2w_1 j}{w_2}) \]

\[ = [2]_{q^n} \sum_{j=0}^{w_1 - 1} (-1)^j q^j \zeta_{p^m, q^n}(w_2 x + \frac{2w_2 j}{w_1}) \]

**Proof.** By substituting \( T_{n, p,q}(x) \) for \( \zeta_{p,q}(s,x) \) in Theorem 7, and using Theorem 6, we can find that

\[ [2]_{q^n} \sum_{j=0}^{w_1 - 1} (-1)^j q^j \zeta_{p^m, q^n}(w_1 x + \frac{2w_1 j}{w_2}) \]

\[ = [2]_{q^n} \sum_{j=0}^{w_1 - 1} (-1)^j q^j T_{n, p^m, q^n}(w_1 x + \frac{2w_1 j}{w_2}) \]

Thus, by (18) and (19), this concludes our proof.
Considering \( w_1 = 1 \) in the Theorem 8, we obtain as below equation.

\[
T_{n,p,q}(x) = \frac{[2]^q_{n^2}}{[2]^q_{p^2}} w_2^n \sum_{j=1}^{w_2-1} (-1)^j q^j T_{n^2,p^2,q^2} \left( \frac{x + 2j}{w_2} \right).
\]

Furthermore, by applying the addition theorem for the Carlitz-type \((h,p,q)\)-tangent polynomials \( T_{n,p,q}(x) \), we can obtain the following theorem.

**Theorem 9.** Let \( w_1 \) and \( w_2 \) be any positive odd integers. Then one has

\[
[2]^q_{n^2} \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_2]^i [w_1]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) = [2]^q_{p^2} \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_2]^i [w_1]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right).
\]

**Proof.** From Theorem 8, we have

\[
[2]^q_{n^2} \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_2]^i [w_1]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) = [2]^q_{p^2} \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_2]^i [w_1]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right).
\]

Therefore, we obtain that

\[
[2]^q_{n^2} [w_2]^n \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_1]_{p,q} [w_2]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) = 0.
\]

and

\[
[2]^q_{p^2} [w_2]^n \sum_{i=0}^{\frac{n}{2}} \binom{n}{i} [w_1]_{p,q} [w_2]_{p,q} [n,l]_{p,q} T_{n^2,p^2,q^2} \left( \frac{w_1}{w_2} \right) T_{n^2,p^2,q^2} \left( \frac{w_2}{w_1} \right) = 0.
\]

where \( T_{n,l,p,q}(k) = \sum_{i=0}^{k-1} (-1)^i q^{(1+2n-2i)} T_{n,l,p,q} \) is called as the alternate power sums. Thus, the theorem can be established by (20) and (21). \( \square \)
5. Zeros of the Carlitz-Type \((P, Q)\)-Tangent Polynomials

The purpose of this section is to support theoretical predictions using numerical experiments and to discover new exciting patterns for zeros of the Carlitz-type \((p, q)\)-tangent polynomials \(T_{n, p,q}(x)\). We propose some conjectures by numerical experiments. The first values of the \(T_{n, p,q}(x)\) are given by

\[
T_{0, p,q}(x) = 1,
\]

\[
T_{1, p,q}(x) = -\frac{-p^x-p^xq^3+q^x+p^2q^{1+x}}{(p-q)(1+p^2q)(1-q+q^2)},
\]

\[
T_{2, p,q}(x) = \frac{p^{2x}+p^{2+2}q^3+p^{2}q^{5}+p^{2+2}q^{8}-2p^{x}q^{x}+q^{2x}-2p^{4+x}q^{1+x}}{(p-q)^2(1+p^2q)(1-p^2q^3)(1-q+q^2)} - \frac{2p^{x}q^{5+x}-2p^{4+x}q^{6+x}+p^{4}q^{1+2x}+p^{2}q^{3+2x}+p^{2}q^{4+2x}}{(1-q+q^2-q^2+q^4)}.
\]

Tables 1 and 2 present the numerical results for approximate solutions of real zeros of \(T_{n, p,q}(x)\). The numbers of zeros of \(T_{n, p,q}(x)\) are tabulated in Table 1 for a fixed \(p = \frac{1}{2}\) and \(q = \frac{1}{10}\).

**Table 1. Numbers of real and complex zeros of** \(T_{n, p,q}(x), p = \frac{1}{2}, q = \frac{1}{10}\).**

<table>
<thead>
<tr>
<th>Degree (n)</th>
<th>Real Zeros</th>
<th>Complex Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>30</td>
<td>2</td>
<td>28</td>
</tr>
</tbody>
</table>

**Table 2. Numerical solutions of** \(T_{n, p,q}(x) = 0, p = \frac{1}{2}, q = \frac{1}{10}\).**

<table>
<thead>
<tr>
<th>Degree (n)</th>
<th>(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0147214</td>
</tr>
<tr>
<td>2</td>
<td>-0.0451666, 0.0490316</td>
</tr>
<tr>
<td>3</td>
<td>0.0737013</td>
</tr>
<tr>
<td>4</td>
<td>-0.0782386, 0.0906197</td>
</tr>
<tr>
<td>5</td>
<td>0.102727</td>
</tr>
<tr>
<td>6</td>
<td>-0.0935042, 0.111767</td>
</tr>
</tbody>
</table>

The use of computer has made it possible to identify the zeros of the Carlitz-type \((p, q)\)-tangent polynomials \(T_{n, p,q}(x)\). The zeros of the Carlitz-type \((p, q)\)-tangent polynomials \(T_{n, p,q}(x)\) for \(x \in \mathbb{C}\) are plotted in Figure 1.

In Figure 1(top-left), we choose \(n = 10, p = 1/2\) and \(q = 1/10\). In Figure 1(top-right), we choose \(n = 20, p = 1/2\) and \(q = 1/10\). In Figure 1(bottom-left), we choose \(n = 30, p = 1/2\) and \(q = 1/10\). In Figure 1(bottom-right), we choose \(n = 40, p = 1/2\) and \(q = 1/10\). It is amazing...
that the structure of the real roots of the Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x)$ is regular. Thus, theoretical prediction on the regular structure of the real roots of the Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x)$ is await for further study (Table 1). Next, we have obtained the numerical solution satisfying Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x) = 0$ for $x \in \mathbb{R}$. The numerical solutions are tabulated in Table 2 for a fixed $p = \frac{1}{2}$ and $q = \frac{1}{10}$ and various value of $n$.

![Figure 1. Zeros of $T_{n,p,q}(x)$.

6. Conclusions and Future Developments

This study constructed the Carlitz-type $(p, q)$-tangent numbers and polynomials. We have derived several formulas for the Carlitz-type $(h, q)$-tangent numbers and polynomials. Some interesting symmetric identities for Carlitz-type $(p, q)$-tangent polynomials are also obtained. Moreover, the results of [18] can be derived from ours as special cases when $q = 1$. By numerical experiments, we will make a series of the following conjectures:

**Conjecture 1.** Prove or disprove that $T_{n,p,q}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry analytic complex functions. Furthermore, $T_{n,p,q}(x)$ has $\text{Re}(x) = a$ reflection symmetry for $a \in \mathbb{R}$.

Many more values of $n$ have been checked. It still remains unknown if the conjecture holds or fails for any value $n$ (see Figure 1).

**Conjecture 2.** Prove or disprove that $T_{n,p,q}(x) = 0$ has $n$ distinct solutions.
In the notations: \( R_{T_{n,p,q}}(x) \) denotes the number of real zeros of \( T_{n,p,q}(x) \) lying on the real plane \( \ln(x) = 0 \) and \( C_{T_{n,p,q}}(x) \) denotes the number of complex zeros of \( T_{n,p,q}(x) \). Since \( n \) is the degree of the polynomial \( T_{n,p,q}(x) \), we get \( R_{T_{n,p,q}}(x) = n - C_{T_{n,p,q}}(x) \) (see Tables 1 and 2).

**Conjecture 3.** Prove or disprove that

\[
R_{T_{n,p,q}}(x) = \begin{cases} 
1, & \text{if } n = \text{odd}, \\
2, & \text{if } n = \text{even}.
\end{cases}
\]

We expect that investigations along these directions will lead to a new approach employing numerical method regarding the research of the Carlitz-type \((p, q)\)-tangent polynomials \( T_{n,p,q}(x) \) which appear in applied mathematics, and mathematical physics (see [11,18–20]).

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**References**


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