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Geometric Properties of Lommel Functions of the First Kind

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Abstract: In the present paper, we find sufficient conditions for starlikeness and convexity of normalized Lommel functions of the first kind using the admissible function methods. Additionally, we investigate some inclusion relationships for various classes associated with the Lommel functions. The functions belonging to these classes are related to the starlike functions, convex functions, close-to-convex functions and quasi-convex functions.

Keywords: Lommel functions; univalent functions; starlike functions; convex functions; inclusion relationships

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathbb{D}$. Also let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are starlike of order α and convex of order α in $\mathbb{D}$, respectively. Analytically, these classes are characterized by the equivalence:

$$f \in \mathcal{S}^*(\alpha) \Longleftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < 1, z \in \mathbb{D})$$

and

$$f \in \mathcal{C}(\alpha) \Longleftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < 1, z \in \mathbb{D}).$$

For convenience, let $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ which are the classes of starlike functions and convex functions, respectively. Furthermore, let $\mathcal{C}(\beta, \alpha)$ and $\mathcal{C}^*(\beta, \alpha)$ be the subclasses of $\mathcal{A}$ defined by

$$\mathcal{C}(\beta,\alpha) = \left\{f \in \mathcal{A} : \exists g \in \mathcal{S}^*(\alpha) \quad \text{s.t.} \quad \Re\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \quad (0 \le \alpha, \beta < 1; z \in \mathbb{D})\right\}$$

and

$$\mathcal{C}^*(\beta,\alpha) = \left\{f \in \mathcal{A} : \exists g \in \mathcal{K}(\alpha) \quad \text{s.t.} \quad \Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \beta \quad (0 \le \alpha, \beta < 1; z \in \mathbb{D})\right\},$$

respectively. The functions in the classes $\mathcal{C}(\beta,\alpha)$ and $\mathcal{C}^*(\beta,\alpha)$ are known as close-to-convex functions and quasi-convex functions, respectively.

The Lommel function of the first kind $s_{\mu,\nu}$ which is expressed in terms of a hypergeometric series

$$s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{z^2}{4}\right),$$

where $\mu\pm\nu$ are not negative odd integers, is a particular solution of the following inhomogeneous Bessel differential equation [1]:

$$z^2w''(z)+zw'(z)+(z^2-\nu^2)w(z)=z^{\mu+1}.$$

It is observed that the function $s_{\mu,\nu}$ does not belong to the class $\mathcal{A}$. Recently, Yağmur [2] and Baricz et al. [3] considered the following function $h_{\mu,\nu}$ defined by:

$$h_{\mu,\nu}(z)=(\mu-\nu+1)(\mu+\nu+1)z^{(1-\mu)/2}s_{\mu,\nu}(\sqrt{z})$$

and they obtained some geometric properties of the function $h_{\mu,\nu}$. For another interesting properties of Lommel function, we can refer to [4,5].

The above function $h_{\mu,\nu}$ belongs to $\mathcal{A}$ and is expressed by:

$$h_{\mu,\nu}(z)=\sum_{n=1}^{\infty}\frac{(-1/4)^{n-1}}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}}z^n\quad((-\mu\pm\nu-3)/2\notin\mathbb{N}:=\{1,2,\cdots\}),\tag{1}$$

where $(\lambda)_n$ is the Pochhammer symbol which defined in terms of Euler's gamma function such that $(\lambda)_n=\Gamma(\lambda+n)/\Gamma(\lambda)=\lambda(\lambda+1)\cdots(\lambda+n-1)$.

Corresponding to the function $h_{\mu,\nu}$ defined by (1), we consider a linear operator $L_{\mu,\nu}:\mathcal{A}\to\mathcal{A}$ defined by:

$$L_{\mu,\nu}f(z)=h_{\mu,\nu}(z)*f(z)\quad((-\mu\pm\nu-3)/2\notin\mathbb{N},z\in\mathbb{D},f\in\mathcal{A}),\tag{2}$$

in terms of the Hadamard product (or convolution) $*$. Then it can be easily observed from (1) and (2) that the following relation holds:

$$z(L_{\mu+1,\nu+1}f(z))'=\left(\frac{\mu+\nu+3}{2}\right)L_{\mu,\nu}f(z)-\left(\frac{\mu+\nu+1}{2}\right)L_{\mu+1,\nu+1}f(z).\tag{3}$$

In a few years ago, many authors introduced new subclasses of univalent (or multivalent) functions by using several linear operators and found many properties of them [6–13]. In [14,15], various inclusion relationships associated with several subclasses of analytic functions were investigated.

Motivated by their works, by using the linear operator $L_{\mu,\nu}$, we define new subclasses of $\mathcal{A}$ as follows:

$$\mathcal{S}^*_{\mu,\nu}(\alpha):=\left\{f\in\mathcal{A}:\Re\left\{\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}f(z)}\right\}>\alpha\;\;(0\le\alpha<1;z\in\mathbb{D})\right\},$$

$$\mathcal{K}_{\mu,\nu}(\alpha):=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{z(L_{\mu,\nu}f(z))''}{(L_{\mu,\nu}f(z))'}\right\}>\alpha\;\;(0\le\alpha<1;z\in\mathbb{D})\right\},$$

$$\mathcal{C}_{\mu,\nu}(\beta,\alpha):=\left\{f\in\mathcal{A}:\exists g\in\mathcal{S}^*_{\mu,\nu}(\alpha)\;\text{ s.t. }\;\Re\left\{\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}g(z)}\right\}>\beta\;\;(0\le\alpha,\beta<1;z\in\mathbb{D})\right\}$$

and

$$\mathcal{C}^*_{\mu,\nu}(\beta,\alpha):=\left\{f\in\mathcal{A}:\exists g\in\mathcal{K}_{\mu,\nu}(\alpha)\;\text{ s.t. }\;\Re\left\{\frac{(z(L_{\mu,\nu}f(z))')'}{(L_{\mu,\nu}g(z))'}\right\}>\beta\;\;(0\le\alpha,\beta<1;z\in\mathbb{D})\right\}.$$

Here, we note that a function f belongs to the class $\mathcal{S}^*_{\mu,\nu}(\alpha)$ ($\mathcal{K}_{\mu,\nu}(\alpha)$, $\mathcal{C}_{\mu,\nu}(\beta,\alpha)$ and $\mathcal{C}^*_{\mu,\nu}(\beta,\alpha)$) is equivalent to that the function $L_{\mu,\nu}f(z)$ belongs to the class $\mathcal{S}^*(\alpha)$ ($\mathcal{K}(\alpha)$, $\mathcal{C}(\beta,\alpha)$ and $\mathcal{C}^*(\beta,\alpha)$, respectively). Further, from the linearity of the operator $L_{\mu,\nu}$, the following relations hold:

$$f(z) \in \mathcal{K}_{\mu,\nu}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*_{\mu,\nu}(\alpha) \tag{4}$$

and

$$f(z) \in \mathcal{C}^*_{\mu,\nu}(\beta,\alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}_{\mu,\nu}(\beta,\alpha). \tag{5}$$

In the present paper some geometric properties of the normalized Lommel function of the first kind are obtained by applying the method of admissible function. In Section 2, we find some sufficient conditions for starlikeness and convexity for the function $h_{\mu,\nu}$. In Section 3, we investigate some inclusion relationships for the classes $\mathcal{S}^*_{\mu,\nu}(\alpha)$, $\mathcal{K}_{\mu,\nu}(\alpha)$, $\mathcal{C}_{\mu,\nu}(\beta,\alpha)$ and $\mathcal{C}^*_{\mu,\nu}(\beta,\alpha)$ which are related to the function $h_{\mu,\nu}$.

The following lemmas will be used for the proof of our results.

Lemma 1. ([16] Miller and Mocanu) *Let Ω be a set in the complex plane $\mathbb{C}$ and let b be a complex number such that $\Re(b) > 0$. Suppose that the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition*

$$\psi(\mathrm{i}\rho, \sigma, a + \mathrm{i}b; z) \notin \Omega$$

for all real $\rho, \sigma, a, b \in \mathbb{R}$ with $\sigma \leq -|b - \mathrm{i}\rho|^2/(2\Re(b))$, $\sigma + a \leq 0$ and $z \in \mathbb{D}$. If the function $p(z)$ defined by $p(z) = b + b_1 z + b_2 z^2 + \ldots$ is analytic in $\mathbb{D}$ and if

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $\Re\{p(z)\} > 0$ in $\mathbb{D}$.

Lemma 2. ([17] Miller and Mocanu) *Let $u = u_1 + \mathrm{i}u_2$, $v = v_1 + \mathrm{i}v_2$ with u_1, u_2, v_1, $v_2 \in \mathbb{R}$ and $\Delta \subset \mathbb{C}^2$. Suppose that $\Phi : \Delta \to \mathbb{C}$ satisfies the following conditions*

1. *$\Phi(u, v)$ is continuous in Δ;*
2. *$(1, 0) \in \Delta$ and $\Re\{\Phi(1, 0)\} > 0$;*
3. *$\Re\{\Phi(\mathrm{i}u_2, v_1)\} \leq 0$ for all $(\mathrm{i}u_2, v_1) \in \Delta$ such that $v_1 \leq -(1 + u_2^2)/2$.*

Let p be an analytic function in $\mathbb{D}$ such that $p(0) = 1$ and $(p(z), zp'(z)) \in \Delta$ for all $z \in \mathbb{D}$. If $\Re\{\Phi(p(z), zp'(z))\} > 0$ in $\mathbb{D}$, then $\Re\{p(z)\} > 0$ in $\mathbb{D}$.

For analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $|\omega(z)| \leq |z|$ such that $f(z) = g(\omega(z))$. Further, if g is univalent, then the definition of subordination $f \prec g$ can be simplified into the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (See [18], p. 36).

Lemma 3. ([19] Eenigenburg et al.) *Let h be convex univalent in $\mathbb{D}$ and w be analytic in $\mathbb{D}$ with $\Re\{w(z)\} \geq 0$ in $\mathbb{D}$. If q is analytic in $\mathbb{D}$ and $q(0) = h(0)$, then the subordination*

$$q(z) + w(z)zq'(z) \prec h(z) \quad (z \in \mathbb{D})$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{D}).$$

Lemma 4. ([2] Yağmur) *If $\mu > -1$, $\nu \in \mathbb{R}$ where $\mu \pm \nu$ are not negative odd integers, and*

$$(\mu+1)[(\mu+1)(\mu+3)-\nu^2] \geq \frac{1}{8},$$

then $\Re\left\{h_{\mu,\nu}(z)/z\right\} > 0$ *in* $\mathbb{D}$.

2. Sufficient Conditions for Starlikeness and Convexity

We find some sufficient conditions for starlikeness and convexity of the function $h_{\mu,\nu}$ given by (1).

Theorem 1. *Let μ and ν be real numbers such that $\mu \pm \nu$ are not negative odd integers, $\mu > 2$,*

$$(\mu+1)[(\mu+1)(\mu+3)-\nu^2] \geq \frac{1}{8} \tag{6}$$

and

$$-\frac{1}{2}(\mu-2)+\frac{1}{96}(\mu-2)^{-1}-\frac{1}{4}\left((\mu-1)^2-\nu^2\right) \leq 0. \tag{7}$$

Then the function $h_{\mu,\nu}$ is a starlike univalent function in $\mathbb{D}$.

Proof. Since

$$h_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1)z^{(1-\mu)/2}s_{\mu,\nu}(\sqrt{z})$$

and the function $s_{\mu,\nu}$ satisfies the inhomogeneous differential equation

$$z^2 s''_{\mu,\nu}(z) + z s'_{\mu,\nu}(z) + (z^2-\nu^2)s_{\mu,\nu}(z) = z^{\mu+1},$$

we have

$$z^2 h''_{\mu,\nu}(z) + \mu z h'_{\mu,\nu}(z) + \frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right)h_{\mu,\nu}(z) - \left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z = 0. \tag{8}$$

Set

$$p(z) = \frac{z h'_{\mu,\nu}(z)}{h_{\mu,\nu}(z)}. \tag{9}$$

From (6) and Lemma 4, $\Re\left\{h_{\mu,\nu}(z)/z\right\} > 0$ for all $z \in \mathbb{D}$ and this implies that $h_{\mu,\nu}(z) \neq 0$ holds for all $z \in \mathbb{D}\setminus\{0\}$. Therefore p is analytic in $\mathbb{D}$ and $p(0) = 1$. Furthermore, by (8) and (9), we have the following equation

$$\left[zp'(z) + p(z)^2 + (\mu-1)p(z) + \frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right)\right]h_{\mu,\nu}(z) = \left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z.$$

Now, we put

$$\tilde{p}(z) = zp'(z) + p(z)^2 + (\mu-1)p(z) + \frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right).$$

Then we have

$$\tilde{p}(z)h_{\mu,\nu}(z) = \left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z.$$

Differentiating the above equation and multiplying by z, we get

$$[z\tilde{p}'(z) + (p(z)-1)\tilde{p}(z)]h_{\mu,\nu}(z) = 0.$$

Since $z\tilde{p}'(z) + (p(z)-1)\tilde{p}(z) = 0$ at $z = 0$ and $h_{\mu,\nu}(z) \neq 0$ for all $z \in \mathbb{D}\setminus\{0\}$, we have

$$z\tilde{p}'(z) + (p(z)-1)\tilde{p}(z) = 0$$

in $\mathbb{D}$, or equivalently,

$$\begin{aligned} &p(z)^3 + (\mu-2)p(z)^2 + z^2p''(z) + 3zp'(z)p(z) + (\mu-1)zp'(z) \\ &+\frac{1}{4}\left(z + (\mu-1)(\mu-5) - \nu^2\right)p(z) - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) = 0 \end{aligned} \tag{10}$$

in $\mathbb{D}$. Now, let $\Omega = \{0\}$ and define a function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\begin{aligned} &\psi(r,s,t;z) \\ &= r^3 + (\mu-2)r^2 + t + 3rs + (\mu-1)s + \frac{1}{4}(z + (\mu-1)(\mu-5) - \nu^2)r - \frac{1}{4}((\mu-1)^2 - \nu^2). \end{aligned}$$

Then the Equation (10) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

Moreover it holds that

$$\begin{aligned} &\Re\{\psi(\rho\mathrm{i}, \sigma, a + \mathrm{i}b; z)\} \\ &= -(\mu-2)\rho^2 + a + (\mu-1)\sigma + \frac{1}{4}\Re\left\{\left(z + (\mu-1)(\mu-5) - \nu^2\right)\mathrm{i}\rho\right\} - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\ &< -\frac{1}{2}(\mu-2)(1 + 3|\rho|^2) + \frac{1}{4}|\rho| - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right), \end{aligned} \tag{11}$$

for $z \in \mathbb{D}$ and $\rho, \sigma, a, b \in \mathbb{R}$ with $\sigma \le -(1+\rho^2)/2$ and $\sigma + a \le 0$. Define a function $g : [0, \infty) \to \mathbb{R}$ by

$$g(\rho) = -\frac{1}{2}(\mu-2)(1+3\rho^2) + \frac{1}{4}\rho.$$

Then, $g'(\rho) = 0$ occurs when $\rho = \rho^* := 1/(12(\mu-2)) > 0$ and $g''(\rho^*) = -3(\mu-2) < 0$. Therefore, the function g has its maximum

$$g(\rho^*) = -\frac{1}{2}(\mu-2) + \frac{1}{96}(\mu-2)^{-1}$$

on the half interval $[0, \infty)$. Hence from (7) and (11) we have

$$\begin{aligned} &\Re\{\psi(\rho\mathrm{i}, \sigma, a + \mathrm{i}b; z)\} \\ &< g(\rho) - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\ &\le -\frac{1}{2}(\mu-2) + \frac{1}{96}(\mu-2)^{-1} - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\ &\le 0, \end{aligned}$$

for all $z \in \mathbb{D}$ and all $\rho, \sigma, a, b \in \mathbb{R}$ with $\sigma \le -(1+\rho^2)/2$ and $\sigma + a \le 0$. By Lemma 1, we have $\Re\{p(z)\} > 0$ in $\mathbb{D}$ which shows that $h_{\mu,\nu}$ is starlike in $\mathbb{D}$. $\square$

Example 1. *We note that $\mu = 5/2$ and $\nu = 1/2$ satisfy the condition of Theorem 1. Therefore the function*

$$h_{5/2,1/2}(z) = 12\left(\frac{z + 2\cos\sqrt{z} - 2}{z}\right) \tag{12}$$

is starlike in $\mathbb{D}$.

Theorem 2. *Let μ and ν be real numbers such that $\mu \pm \nu$ are not negative odd integers, $\mu > 2$,*

$$(\mu+1)[(\mu+1)(\mu+3)-\nu^2] \geq \frac{1}{8}$$

and

$$\begin{cases} \frac{1}{96}(\mu-2)^{-1}+(\mu-2)-\frac{1}{4}\left((\mu-1)^2-\nu^2\right) \leq 0, & \text{if } \mu \leq \frac{25}{12}, \\ -\frac{1}{2}\mu+\frac{5}{4}-\frac{1}{4}\left((\mu-1)^2-\nu^2\right) \leq 0, & \text{if } \mu > \frac{25}{12}. \end{cases} \tag{13}$$

Then the function $h_{\mu,\nu}$ is a convex univalent function in $\mathbb{D}$.

Proof. First of all, we observe that the condition (13) implies (7) in Theorem 1. To see this, we assume that the inequality (13) holds. For the case $2 < \mu \leq 25/12$, from the inequality $-(\mu-2)/2 \leq \mu - 2$, we can easily obtain the inequality (7). For the case $\mu > 25/12$, it is sufficient to check the following inequality holds:

$$-\frac{1}{2}(\mu-2)^2+\frac{1}{96} \leq -\frac{1}{2}\mu(\mu-2)+\frac{5}{4}(\mu-2).$$

And the above inequality is true for $\mu > 25/12$, since

$$-\frac{1}{2}\mu(\mu-2)+\frac{5}{4}(\mu-2)+\frac{1}{2}(\mu-2)^2-\frac{1}{96} = \frac{1}{4}\left(\mu-\frac{49}{24}\right) > \frac{1}{96}.$$

Therefore the function $h_{\mu,\nu}$ is starlike univalent, hence $h'_{\mu,\nu}(z) \neq 0$ in $\mathbb{D}$. Now, set

$$p(z) = 1+\frac{zh''_{\mu,\nu}(z)}{h'_{\mu,\nu}(z)} \quad (z \in \mathbb{D}).$$

Since $h'_{\mu,\nu}(z) \neq 0$ in $\mathbb{D}$, p is analytic in $\mathbb{D}$ with $p(0) = 1$. And we have

$$zh''_{\mu,\nu}(z) = (p(z)-1)h'_{\mu,\nu}(z) \tag{14}$$

and

$$2zh''_{\mu,\nu}(z)+z^2h^{(3)}_{\mu,\nu}(z) = [zp'(z)+p(z)^2-p(z)]h'_{\mu,\nu}(z). \tag{15}$$

Furthermore, from (8), we have

$$(p(z)+\mu-1)zh'_{\mu,\nu}(z)+\frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right)h_{\mu,\nu}(z)-\left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z = 0. \tag{16}$$

Differentiating (16) and multiplying by z, we get

$$\begin{aligned} &z^2p'(z)h'_{\mu,\nu}(z)+(p(z)+\mu-1)zh'_{\mu,\nu}(z)+(p(z)+\mu-1)z^2h''_{\mu,\nu}(z) \\ &+\frac{1}{4}zh_{\mu,\nu}(z)+\frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right)zh'_{\mu,\nu}(z) \\ &-\left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z = 0. \end{aligned} \tag{17}$$

Substituting (17) into (16), we obtain

$$\begin{aligned} &(p(z)+\mu-1)z^2h''_{\mu,\nu}(z)+[zp'(z)+\frac{1}{4}\left(z+(\mu-1)^2-\nu^2\right)]zh'_{\mu,\nu}(z) \\ &-\frac{1}{4}\left((\mu-1)^2-\nu^2\right)h_{\mu,\nu}(z) = 0. \end{aligned} \tag{18}$$

Differentiating (18) and using the equalities (14) and (15), we get

$$
\begin{aligned}
&z^2 p''(z) + 3zp'(z)p(z) + (\mu-1)zp'(z) + p(z)^3 + (\mu-2)p(z)^2 \\
&+\frac{1}{4}\left(z + (\mu-1)(\mu-5) - \nu^2\right) p(z) + \frac{1}{4}\left(z - (\mu-1)^2 + \nu^2\right) = 0.
\end{aligned} \tag{19}
$$

Now, let $\Omega = \{0\}$ and define a function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$
\begin{aligned}
&\psi(r,s,t;z) \\
&= t + 3rs + (\mu-1)s + r^3 + (\mu-2)r^2 + \frac{1}{4}\left(z + (\mu-1)(\mu-5) - \nu^2\right) r + \frac{1}{4}\left(z - (\mu-1)^2 + \nu^2\right).
\end{aligned}
$$

Then, (19) becomes

$$
\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega.
$$

And simple calculations give us that

$$
\begin{aligned}
&\Re\{\psi(\mathrm{i}\rho, \sigma, a + \mathrm{i}b; z)\} \\
&= a + (\mu-1)\sigma - (\mu-2)\rho^2 + \frac{1}{4}\Re\{\mathrm{i}\rho z\} + \frac{1}{4}\Re\left\{z - (\mu-1)^2 + \nu^2\right\} \\
&\leq -(\mu-2)\rho^2 + (\mu-2)\sigma + \frac{1}{4}\Re\{(1+\mathrm{i}\rho)z\} - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&< -(\mu-2)\rho^2 - \frac{1}{2}(\mu-2)(1+\rho^2) + \frac{1}{4}\sqrt{1+\rho^2} - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&= -\frac{3}{2}(\mu-2)u^2 + (\mu-2) + \frac{1}{4}u - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right),
\end{aligned} \tag{20}
$$

for all $z \in \mathbb{D}$ and all ρ, σ, a, b, $u \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$, $\sigma + a \leq 0$ and $u = \sqrt{1+\rho^2}$. Define a function $g : [1, \infty) \to \mathbb{R}$ by

$$
g(u) = -\frac{3}{2}(\mu-2)u^2 + \frac{1}{4}u + \mu - 2.
$$

Then, by putting $u^* = 1/(12(\mu-2)) > 0$, we have $g'(u^*) = 0$. Moreover it holds that $g''(u) = -3(\mu-2) < 0$ for all $u \in [1, \infty)$. Therefore $u = u^*$ gives the maximum value for g when $\mu \leq 25/12$. On the other hand, when $\mu > 25/12$ the function g is maximized by setting $u = 1$. Hence, for the case $\mu \leq 25/12$, it follows from (13) and (20) that

$$
\begin{aligned}
&\Re\{\psi(\mathrm{i}\rho, \sigma, a + \mathrm{i}b; z)\} \\
&< g(u^*) - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&\leq \frac{1}{96}(\mu-2)^{-1} + (\mu-2) - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&\leq 0,
\end{aligned}
$$

for all $z \in \mathbb{D}$ and all ρ, σ, a, $b \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$ and $\sigma + a \leq 0$. Similarly, for the case $\mu > 25/12$, we obtain

$$
\begin{aligned}
&\Re\{\psi(\mathrm{i}\rho, \sigma, a + \mathrm{i}b; z)\} \\
&< g(1) - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&\leq -\frac{1}{2}\mu + \frac{5}{4} - \frac{1}{4}\left((\mu-1)^2 - \nu^2\right) \\
&\leq 0,
\end{aligned}
$$

for all $z \in \mathbb{D}$ and all ρ, σ, a, $b \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$ and $\sigma + a \leq 0$. By Lemma 1, we thus have $\Re\{p(z)\} > 0$ in $\mathbb{D}$ which shows that $h_{\mu,\nu}$ is convex in $\mathbb{D}$. $\square$

Example 2. *We note that* $\mu = 5/2$ *and* $\nu = 1/2$ *satisfy the condition of Theorem 2. Therefore the function* $h_{5/2,1/2}$ *given by* (12) *is convex in* $\mathbb{D}$.

3. Inclusion Relationships

Now, we investigate some inclusion relationships for the classes $\mathcal{S}^*_{\mu,\nu}(\alpha)$, $\mathcal{K}_{\mu,\nu}(\alpha)$, $\mathcal{C}_{\mu,\nu}(\beta,\alpha)$ and $\mathcal{C}^*_{\mu,\nu}(\beta,\alpha)$. We begin by proving our first inclusion relationship for the class $\mathcal{S}^*_{\mu,\nu}(\alpha)$.

Theorem 3. *Let* μ, ν *and* α *be real numbers such that* $\mu \pm \nu$ *are not negative odd integers,* $0 \le \alpha < 1$ *and* $2\alpha + \mu + \nu + 1 \ge 0$. *Then*

$$\mathcal{S}^*_{\mu,\nu}(\alpha) \subset \mathcal{S}^*_{\mu+1,\nu+1}(\alpha).$$

Proof. Let $f \in \mathcal{S}^*_{\mu,\nu}(\alpha)$ and define a function $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = \frac{1}{1-\alpha}\left(\frac{z(L_{\mu+1,\nu+1}f(z))'}{L_{\mu+1,\nu+1}f(z)} - \alpha\right). \tag{21}$$

Then ϕ is analytic in $\mathbb{D}$ and $\phi(0) = 1$. From the equality (3), we get

$$\left(\frac{\mu+\nu+3}{2}\right)\frac{L_{\mu,\nu}f(z)}{L_{\mu+1,\nu+1}f(z)} = \frac{z(L_{\mu+1,\nu+1}f(z))'}{L_{\mu+1,\nu+1}f(z)} + \frac{\mu+\nu+1}{2}. \tag{22}$$

By combining (21) and (22), we obtain

$$\frac{L_{\mu,\nu}f(z)}{L_{\mu+1,\nu+1}f(z)} = \frac{2}{\mu+\nu+3}\left[(1-\alpha)\phi(z) + \alpha + \frac{\mu+\nu+1}{2}\right]. \tag{23}$$

Now, by applying the logarithmic differentiation on both sides of (23) and multiplying the resulting equation by z, we have

$$\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}f(z)} = \frac{z(L_{\mu+1,\nu+1}f(z))'}{L_{\mu+1,\nu+1}f(z)} + \frac{(1-\alpha)z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \frac{\mu+\nu+1}{2}}$$

which, in view of (21), yields

$$\frac{1}{1-\alpha}\left(\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}f(z)} - \alpha\right) = \phi(z) + \frac{z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \frac{\mu+\nu+1}{2}}. \tag{24}$$

Now, we define a function $\Phi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\Phi(u,v) = u + \frac{v}{(1-\alpha)u + \alpha + \frac{\mu+\nu+1}{2}}.$$

Observe that Φ is continuous on

$$\Delta := \left(\mathbb{C} \setminus \left\{\frac{\alpha + \frac{\mu+\nu+1}{2}}{\alpha - 1}\right\}\right) \times \mathbb{C},$$

$(1,0) \in \Delta$ and $\Re\{\Phi(1,0)\} > 0$. Since $f \in \mathcal{S}^*_{\mu,\nu}(\alpha)$, it follows from (24) that $\Re\{\Phi(\phi(z), z\phi'(z), z^2\phi''(z))\} > 0$ for all $z \in \mathbb{D}$. Also, for $(\mathrm{i}u_2, v_1) \in \Delta$ with u_2, $v_1 \in \mathbb{R}$ such that $v_1 \le -(1+u_2^2)/2$, we have

$$\begin{aligned}\Re\left\{\Phi(\mathrm{i}u_2, v_1)\right\} &= \Re\left\{\mathrm{i}u_2 + \frac{v_1}{\mathrm{i}(1-\alpha)u_2 + \alpha + \frac{\mu+\nu+1}{2}}\right\} \\ &= \frac{v_1\left(\alpha + \frac{\mu+\nu+1}{2}\right)}{(1-\alpha)^2 u_2^2 + \left(\alpha + \frac{\mu+\nu+1}{2}\right)^2} \\ &\leq -\frac{1}{2}\left(1+u_2^2\right)\frac{\alpha + \frac{\mu+\nu+1}{2}}{(1-\alpha)^2 u_2^2 + \left(\alpha + \frac{\mu+\nu+1}{2}\right)^2} \\ &< 0\end{aligned}$$

which shows that $\Re\left\{\Phi(\mathrm{i}u_2, v_1)\right\} < 0$. Therefore, by Lemma 2, we have

$$\Re\left\{\phi(z)\right\} > 0 \quad (z \in \mathbb{D}).$$

Thus, by making use of (21), we find that $f \in \mathcal{S}^*_{\mu+1,\nu+1}(\alpha)$. This completes the proof of Theorem 3. □

Theorem 4. *Let μ, ν and α be real numbers such that $\mu \pm \nu$ are not negative odd integers, $0 \leq \alpha < 1$ and $2\alpha + \mu + \nu + 1 \geq 0$. Then*

$$\mathcal{K}_{\mu,\nu}(\alpha) \subset \mathcal{K}_{\mu+1,\nu+1}(\alpha).$$

Proof. By applying (4) and Theorem 3, we observe that

$$\begin{aligned} f \in \mathcal{K}_{\mu,\nu}(\alpha) &\Longleftrightarrow zf' \in \mathcal{S}^*_{\mu,\nu}(\alpha) \\ &\Longrightarrow zf' \in \mathcal{S}^*_{\mu+1,\nu+1}(\alpha) \\ &\Longleftrightarrow f \in \mathcal{K}_{\mu+1,\nu+1}(\alpha)\end{aligned}$$

which proves Theorem 4. □

Theorem 5. *Let μ, ν, α and β be real numbers such that $\mu \pm \nu$ are not negative odd integers, $0 \leq \alpha < 1$, $0 \leq \beta < 1$ and $2\alpha + \mu + \nu + 1 \geq 0$. Then*

$$\mathcal{C}_{\mu,\nu}(\beta, \alpha) \subset \mathcal{C}_{\mu+1,\nu+1}(\beta, \alpha).$$

Proof. Let $f \in \mathcal{C}_{\mu,\nu}(\beta, \alpha)$. Then there exists a function $g \in \mathcal{S}^*_{\mu,\nu}(\alpha)$ such that

$$\Re\left\{\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}g(z)}\right\} > \beta. \tag{25}$$

Define a function $\phi : \mathbb{D} \to \mathbb{C}$ by

$$\phi(z) = \frac{1}{1-\beta}\left(\frac{z(L_{\mu+1,\nu+1}f(z))'}{L_{\mu+1,\nu+1}g(z)} - \beta\right). \tag{26}$$

Then, ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. Using the identity (3), we also have

$$\begin{aligned}
\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}g(z)} &= \frac{L_{\mu,\nu}(zf'(z))}{L_{\mu,\nu}g(z)} \\
&= \frac{z(L_{\mu+1,\nu+1}(zf'(z)))' + \left(\frac{\mu+\nu+1}{2}\right) L_{\mu+1,\nu+1}(zf'(z))}{z(L_{\mu+1,\nu+1}g(z))' + \left(\frac{\mu+\nu+1}{2}\right) L_{\mu+1,\nu+1}g(z)} \\
&= \frac{\frac{z(L_{\mu+1,\nu+1}(zf'(z)))'}{L_{\mu+1,\nu+1}g(z)} + \left(\frac{\mu+\nu+1}{2}\right) \frac{L_{\mu+1,\nu+1}(zf'(z))}{L_{\mu+1,\nu+1}g(z)}}{\frac{z(L_{\mu+1,\nu+1}g(z))'}{L_{\mu+1,\nu+1}g(z)} + \frac{\mu+\nu+1}{2}}.
\end{aligned} \tag{27}$$

Now we define a function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1}{1-\alpha}\left(\frac{z(L_{\mu+1,\nu+1}g(z))'}{L_{\mu+1,\nu+1}g(z)} - \alpha\right). \tag{28}$$

Since $g \in \mathcal{S}^*_{\mu,\nu}(\alpha)$, by Theorem 3, we have $g \in \mathcal{S}^*_{\mu+1,\nu+1}(\alpha)$ and therefore we get $\Re\{q(z)\} > 0$ in $\mathbb{D}$. Upon substituting from (26) and (28) into (27), we have

$$\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}g(z)} = \frac{\frac{z(L_{\mu+1,\nu+1}(zf'(z)))'}{L_{\mu+1,\nu+1}g(z)} + \left(\frac{\mu+\nu+1}{2}\right)((1-\beta)\phi(z)+\beta)}{(1-\alpha)q(z)+\alpha+\frac{\mu+\nu+1}{2}}. \tag{29}$$

By logarithmically differentiating both sides of (26) with respect to z, we have

$$\frac{z(L_{\mu+1,\nu+1}(zf'(z)))'}{L_{\mu+1,\nu+1}g(z)} = ((1-\beta)\phi(z)+\beta)((1-\alpha)q(z)+\alpha) + (1-\beta)z\phi'(z)$$

which, in conjunction with (29), yields

$$\frac{1}{1-\beta}\left(\frac{z(L_{\mu,\nu}f(z))'}{L_{\mu,\nu}g(z)} - \beta\right) = \phi(z) + \frac{z\phi'(z)}{(1-\alpha)q(z)+\alpha+\frac{\mu+\nu+1}{2}}.$$

Put

$$\omega(z) = \frac{1}{(1-\alpha)q(z)+\alpha+\frac{\mu+\nu+1}{2}}.$$

Then, ω is analytic in $\mathbb{D}$ and, from the inequality (25), we have

$$\Re\{\phi(z) + \omega(z)z\phi'(z)\} > 0$$

in $\mathbb{D}$. Using the fact that $\Re\{q(z)\} > 0$ in $\mathbb{D}$ and the inequality $2\alpha+\mu+\nu+1 \geq 0$, we have $\Re\{\omega(z)\} > 0$ in $\mathbb{D}$. Applying Lemma 3 with $h(z) = (1+z)/(1-z)$, we have $\Re\{\phi(z)\} > 0$ in $\mathbb{D}$. Thus, by making use of (26), we get $f \in \mathcal{C}_{\mu+1,\nu+1}(\beta,\alpha)$. This completes the proof of Theorem 5. □

Finally, we state the inclusion relationship for the class $\mathcal{C}^*_{\mu,\nu}(\beta,\alpha)$.

Theorem 6. *Let μ, ν, α and β be real numbers such that $\mu \pm \nu$ are not negative odd integers, $0 \leq \alpha < 1$, $0 \leq \beta < 1$ and $2\alpha+\mu+\nu+1 \geq 0$. Then*

$$\mathcal{C}^*_{\mu,\nu}(\beta,\alpha) \subset \mathcal{C}^*_{\mu+1,\nu+1}(\beta,\alpha).$$

Proof. By applying (5) and Theorem 5, we observe that

$$\begin{aligned}
f \in \mathcal{C}^*_{\mu,\nu}(\beta,\alpha) &\Longleftrightarrow zf'(z) \in \mathcal{C}_{\mu,\nu}(\beta,\alpha) \\
&\Longrightarrow zf'(z) \in \mathcal{C}_{\mu+1,\nu+1}(\beta,\alpha) \\
&\Longleftrightarrow f \in \mathcal{C}^*_{\mu+1,\nu+1}(\beta,\alpha)
\end{aligned}$$

which proves Theorem 6. □

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