Pythagorean Fuzzy Hamy Mean Operators in Multiple Attribute Group Decision Making and Their Application to Supplier Selection

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Abstract: In this paper, we extend the Hamy mean (HM) operator and dual Hamy mean (DHM) operator with Pythagorean fuzzy numbers (PFNs) to propose Pythagorean fuzzy Hamy mean (PFHM) operator, weighted Pythagorean fuzzy Hamy mean (WPFHM) operator, Pythagorean fuzzy dual Hamy mean (PFDHM) operator, weighted Pythagorean fuzzy dual Hamy mean (WPFDHM) operator. Then the multiple attribute group decision making (MAGDM) methods are proposed with these operators. In the end, we utilize an applicable example for supplier selection to prove the proposed methods.

Keywords: multiple attribute group decision making (MAGDM); Pythagorean fuzzy numbers; Pythagorean fuzzy sets (PFSs); PFHM operator; WPFHM operator; PFDHM operator; WPFDHM operator; supplier selection

1. Introduction

hesitant PFNs and Wei et al. [44] proposed some Pythagorean hesitant fuzzy hamacher operators based on the traditional hamacher operators [45–47]. Wei et al. [48], Tang and Wei [49] and Huang and Wei [50] proposed the Pythagorean 2-tuple linguistic operators in MADM with the traditional arithmetic and geometric operators [51–56]. Wei et al. [57] proposed some q-Rung Orthopair fuzzy Heronian mean operators in MADM.

And HM operator [58] and DHM operator [59] are famous operators which depict interrelationships among any number of arguments assigned by a variable vector. Therefore, the HM and DHM operators can furnish a robust and flexible mechanism to solve the information fusion in MAGDM problems. Because PFNs can easily capture the fuzzy information and the HM can describe interrelationships among any number of arguments assigned by a variable vector, it is necessary to expand the HM and DHM operators to deal with the PFNs. Thus, how to fuse these PFNs with HM and DHM operators is an interesting topic. In order to accomplish this goal, the remainder of this paper is arranged as follows. In the next section, we introduce some basic concepts related to PFNs. In Section 3, we propose some HM and DHM operators with PFNs. In Section 4, we present some methods for MAGDM problems with PFWHM and PFWDHM operator. In Section 5, we give a numerical example. Finally, a brief conclusion is given in Section 6.

2. Preliminaries

2.1. Pythagorean Fuzzy Sets

Yager [1,2] gave the definition of PFSs.

**Definition 1.** Let $X$ be a fixed set. A PFS in $X$ is defined as follows [1,2]

$$P = \{(x, (\mu_p(x), \nu_p(x))) | x \in X\}$$

(1)

where $\mu_p(x) \in [0,1]$ and $\nu_p(x) \in [0,1]$ are defined as the degree of membership and non-membership of the element $x \in X$ to $P$, respectively, and satisfying

$$(\mu_p(x))^2 + (\nu_p(x))^2 \leq 1.$$  

(2)

**Definition 2.** Let $\tilde{a} = (\mu, \nu)$ be a PFN, then the score function of $\tilde{a}$ is defined [17]

$$S(\tilde{a}) = \frac{1}{2} (1 + \mu^2 - \nu^2), S(\tilde{a}) \in [0,1].$$

(3)

**Definition 3.** Let $\tilde{a} = (\mu, \nu)$ be a PFN, then the accuracy function of $\tilde{a}$ is defined [11]

$$H(\tilde{a}) = \mu^2 + \nu^2, H(\tilde{a}) \in [0,1].$$

(4)

**Definition 4.** Let $\tilde{a}_1 = (\mu_1, v_1)$ and $\tilde{a}_2 = (\mu_2, v_2)$ be two PFNs [17], $S(\tilde{a}_1) = \frac{1}{2} (1 + (\mu_1)^2 - (v_2)^2)$ and

$S(\tilde{a}_2) = \frac{1}{2} (1 + (\mu_2)^2 - (v_2)^2)$ be the scores function of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and let $H(\tilde{a}_1) = \mu_1^2 + v_1^2$ and $H(\tilde{a}_2) = \mu_2^2 + v_2^2$ be the accuracy function of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, then if $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1 < \tilde{a}_2$ if $S(\tilde{a}_1) = S(\tilde{a}_2)$, then

1. If $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$;
2. If $H(\tilde{a}_1) < H(\tilde{a}_2)$, then $\tilde{a}_1 < \tilde{a}_2$. 


Example 1. Let $\tilde{a}_1 = (0.5, 0.3), \tilde{a}_2 = (0.6, 0.2), \tilde{a}_3 = (0.4, 0)$, according to Definitions 2–4, we get

$$S(\tilde{a}_1) = \frac{1}{2} (1 + 0.5^2 - 0.3^2) = 0.5800, S(\tilde{a}_2) = \frac{1}{2} (1 + 0.6^2 - 0.2^2) = 0.6600$$

$$S(\tilde{a}_3) = \frac{1}{2} (1 + 0.4^2 - 0^2) = 0.5800, H(\tilde{a}_1) = 0.5^2 + 0.3^2 = 0.3400$$

$$H(\tilde{a}_2) = 0.6^2 + 0.2^2 = 0.4000, H(\tilde{a}_3) = 0.4^2 + 0^2 = 0.1600$$

Then we can conclude that $\tilde{a}_2 > \tilde{a}_1 > \tilde{a}_3$.

Definition 5. Let $\tilde{a}_1 = (\mu_1, \nu_1), \tilde{a}_2 = (\mu_2, \nu_2)$ and $\tilde{a} = (\mu, \nu)$ be three PFNs, and some basic operations are defined [5]:

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \left( \sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2 (\mu_2)^2}, \nu_1 \nu_2 \right)$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \left( \mu_1 \mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2 (\nu_2)^2} \right)$;
3. $\lambda \tilde{a} = \left( \sqrt{1 - (1 - \mu)^\lambda}, \nu \right), \lambda > 0$;
4. $(\tilde{a})^\lambda = \left( \mu^\lambda, \sqrt{1 - (1 - \nu)^\lambda} \right), \lambda > 0$;
5. $\tilde{a}^c = (\nu, \mu)$.

Example 2. Suppose that $\tilde{a}_1 = (0.2, 0.3), \tilde{a}_2 = (0.6, 0.1)$, and $\lambda = 5$, then we have

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \left( \sqrt{0.2^2 + 0.6^2 - 0.2^2 \times 0.6^2}, 0.3 \times 0.1 \right) = (0.5456, 0.0300)$
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \left( 0.2 \times 0.6, \sqrt{0.3^2 + 0.1^2 - 0.3^2 \times 0.1^2} \right) = (0.1200, 0.3091)$
3. $\lambda \tilde{a}_1 = \left( \sqrt{1 - (1 - 0.2)^{0.5}}, 0.3^{0.5} \right) = (0.1056, 0.5477)$
4. $(\tilde{a}_1)^\lambda = \left( 0.2^{0.5}, \sqrt{1 - (1 - 0.3)^{0.5}} \right) = (0.4472, 0.1633)$
5. $\tilde{a}_1^c = (0.3, 0.2)$

The following properties are derived based on the Definition 4.

Definition 6. Let $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ be two PFNs, $\lambda, \lambda_1, \lambda_2 > 0$, then [5]

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$;
3. $\lambda(\tilde{a}_1 \oplus \tilde{a}_2) = \lambda \tilde{a}_1 + \lambda \tilde{a}_2$;
4. $(\tilde{a}_1 \cdot \tilde{a}_2)^\lambda = (\tilde{a}_1)^\lambda \cdot (\tilde{a}_2)^\lambda$;
5. $\lambda_1 \tilde{a}_1 \oplus \lambda_2 \tilde{a}_1 = (\lambda_1 + \lambda_2) \tilde{a}_1$;
6. $(\tilde{a}_1)^{\lambda_1} \cdot (\tilde{a}_2)^{\lambda_2} = (\tilde{a}_1)^{\lambda_1 + \lambda_2}$;
7. $\left( (\tilde{a}_1)^{\lambda_1} \right)^{\lambda_2} = (\tilde{a}_1)^{\lambda_1 \lambda_2}$. 
2.2. HM Operator

**Definition 7.** The HM operator is defined as follows \[58]:

\[
\text{HM}^{(x)}(a_1, a_2, \ldots, a_k) = \frac{1}{\sum_{i=1}^{k} a_i} \left( \prod_{j=1}^{x} a_j \right)^{\frac{1}{x}}
\]

where \(x\) is a parameter and \(x = 1, 2, \ldots, k, i_1, i_2, \ldots, i_x\) are \(x\) integer values taken from the set \(\{1, 2, \ldots, k\}\) of \(k\) integer values, \(C^x_k\) denotes the binomial coefficient and \(C^x_k = \frac{k!}{(k-x)!x!}\).

The HM operator has three properties.

(i) when \(a_i = a (i = 1, 2, \ldots, k)\), \(\text{HM}^{(x)}(a_1, a_2, \ldots, a_k) = a\);  
(ii) when \(a_i \leq \pi_j (i = 1, 2, \ldots, k)\), \(\text{HM}^{(x)}(a_1, a_2, \ldots, a_k) \leq \text{HM}^{(x)}(\pi_1, \pi_2, \ldots, \pi_k)\);  
(iii) when \(\min\{a_i\} \leq \text{HM}^{(x)}(a_1, a_2, \ldots, a_k) \leq \max\{a_i\}\).

Two particular cases of the HM operator are given as follows.

(1) When \(x = 1\), \(\text{HM}^{(1)}(a_1, a_2, \ldots, a_k) = \frac{1}{k} \sum_{i=1}^{k} a_i\), it becomes the arithmetic mean operator.  
(2) When \(x = k\), \(\text{HM}^{(k)}(a_1, a_2, \ldots, a_k) = \left( \prod_{j=1}^{k} a_j \right)^{\frac{1}{k}}\), it becomes the geometric mean operator.

3. The HM Operators for PFNs

In this part, we will combine PFNs and HM operator, and propose the PFHM operator and WPFHM operator.

### 3.1. The PFHM Operator

**Definition 8.** Let \(\tilde{a}_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, k)\) be a set of PFNs, then we can define PFHM operator as follows:

\[
\text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \frac{\sum_{i=1}^{k} \tilde{a}_i}{C^x_k} \left( \prod_{j=1}^{x} \tilde{a}_j \right)^{\frac{1}{x}}
\]

where \(x\) is a parameter and \(x = 1, 2, \ldots, k, i_1, i_2, \ldots, i_x\) are \(x\) integer values taken from the set \(\{1, 2, \ldots, k\}\) of \(k\) integer values, \(C^x_k\) denotes the binomial coefficient and \(C^x_k = \frac{k!}{(k-x)!x!}\).

**Theorem 1.** Let \(\tilde{a}_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, k)\) be a set of the PFNs, then the aggregate result of Definition 8 is still a PFNs, and have

\[
\text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \frac{1}{\sum_{i=1}^{k} \tilde{a}_i} \left( \prod_{j=1}^{x} \tilde{a}_j \right)^{\frac{1}{x}}
\]

\[
= \left( 1 - \left( \prod_{1 \leq i_1 < \cdots < i_x < k} \left( 1 - \left( \prod_{j=1}^{x} \left( \mu_{j} \right)^{\frac{1}{x}} \right) \right) \right) \right) \left( \prod_{1 \leq i_1 < \cdots < i_x < k} \left( 1 - \left( \prod_{j=1}^{x} \left( \frac{1}{x} \left( 1 - (\nu_j)^{2} \right) \right) \right) \right) \right)^{\frac{1}{x}}
\]
Proof.

(1) First of all, we prove (7) is kept.

\[ x_j \otimes \bar{a}_{ij} = \left( \prod_{j=1}^{x} (\mu_i) \right) \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_{ij})^2\right)} \]  

(8)

Therefore,

\[ \left( \prod_{j=1}^{x} \bar{a}_{ij} \right) = \left( \prod_{j=1}^{x} (\mu_i) \right) \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_{ij})^2\right)} \]  

(9)

Moreover,

\[ \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left( \prod_{j=1}^{x} (\mu_i) \right) \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_{ij})^2\right)} \]  

(10)

Furthermore,

\[ \text{PFHM}^{(s)}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left( \prod_{j=1}^{x} (\mu_i) \right) \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_{ij})^2\right)} \]  

(11)

(2) Next, we prove (7) is a PFN. Let

\[ p = \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left(1 - \left( \prod_{j=1}^{x} (\mu_i) \right) \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_{ij})^2\right)}\right)^{\frac{1}{k}} \]  

\[ q = \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left(1 - \left( \prod_{j=1}^{x} (1 - (v_{ij})^2)\right)\right)^{\frac{1}{k}} \]  

Then we prove the following two conditions. (i) \(0 \leq p \leq 1, 0 \leq q \leq 1;\) (ii) \(0 \leq p^2 + q^2 \leq 1.

i. Since \(\mu_{ij} \in [0, 1],\) we can get

\[ \prod_{j=1}^{x} \mu_{ij} \in [0, 1] \Rightarrow \left( \prod_{j=1}^{x} (\mu_i) \right) \in [0, 1] \Rightarrow \left( \prod_{j=1}^{x} (\mu_i) \right)^2 \in [0, 1] \Rightarrow 1 - \left( \prod_{j=1}^{x} (\mu_i) \right)^2 \in [0, 1] \]

\[ \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left(1 - \left( \prod_{j=1}^{x} (\mu_i) \right)^2\right) \in [0, 1] \Rightarrow \prod_{1 \leq i_1 < \cdots < i_k \leq x} \left(1 - \left( \prod_{j=1}^{x} (1 - (v_{ij})^2)\right)^2\right) \in [0, 1] \]
\[ \Rightarrow \sqrt{1 - \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \mu_{i_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \right) \in [0, 1]} \text{ i.e., } 0 \leq p \leq 1. \]

Similarly, we can get
\[ \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{i_j})^2) \right)^{\frac{1}{q}} \right) \in [0, 1], \text{i.e., } 0 \leq q \leq 1. \]

ii. Obviously, \( 0 \leq p^2 + q^2 \leq 1 \), then
\[
\left( \sqrt{1 - \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \mu_{i_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \right) \right)^2 + \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{i_j})^2) \right)^{\frac{1}{q}} \right) \right)^2 \leq 1.
\]

We get \( 0 \leq p^2 + q^2 \leq 1. \)

So the aggregated result of Definition 8 is still PFN. Next we will talk about some properties of PFHM operator.

**Property 1.** (Idempotency). If \( \tilde{a}_i(1, 2, \cdots, k) \) and \( \tilde{a} \) are PFNs, and \( \tilde{a}_i = \tilde{a} = (\mu_i, v_i) \) for all \( i = 1, 2, \cdots, k \), then we get
\[ \text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \tilde{a} \tag{12} \]

**Proof.** Since \( \tilde{a} = (\mu, v) \), based on Theorem 1, we have
\[
\text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left( \sqrt{1 - \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \mu_{i_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \right) \right)^2 + \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{i_j})^2) \right)^{\frac{1}{q}} \right) \right)^2 \leq 1.
\]

\( \Box \)

**Property 2.** (Monotonicity). Let \( \tilde{a}_i = (\mu_{i_j}, v_{i_j}), \tilde{a}_i = (\mu_{i_j}, v_{i_j}) \) \((i = 1, 2, \cdots, k) \) be two sets of PFNs. If \( \mu_{i_j} \geq \mu_{\theta_i}, v_{i_j} \leq v_{\theta_j} \) for all \( j \), then
\[ \text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \geq \text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \tag{13} \]

\( \Box \)
Proof. Since \( x \geq 1, \mu_i \geq \mu_0, v_i \geq 0, v_0 \geq v_j \geq 0 \), then

\[
\left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \geq \left( \prod_{i=1}^{x} \mu_0 \right)^{\frac{1}{2}} \Rightarrow 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \leq 1 - \left( \prod_{i=1}^{x} \mu_0 \right)^{\frac{1}{2}}
\]

\[
\Rightarrow \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}} \leq \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_0 \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}}
\]

\[
\Rightarrow \sqrt{1 - \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}}} \leq \sqrt{1 - \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_0 \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}}}
\]

Similarly, we have

\[
1 - (v_j)^2 \geq 1 - (\pi \theta_j)^2 \Rightarrow \prod_{j=1}^{x} \left( 1 - (v_j)^2 \right) \leq \prod_{j=1}^{x} \left( 1 - (\pi \theta_j)^2 \right)
\]

\[
\Rightarrow \prod_{1 \leq i_1 \cdots < i_k \leq k} \left[ 1 - \prod_{j=1}^{x} \left( 1 - (v_j)^2 \right) \right]^{\frac{1}{k}} \leq \prod_{1 \leq i_1 \cdots < i_k \leq k} \left[ 1 - \prod_{j=1}^{x} \left( 1 - (\pi \theta_j)^2 \right) \right]^{\frac{1}{k}}
\]

Let \( \tilde{a} = \text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \), \( \tilde{\pi} = \text{PFHM}^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k) \) and \( S(\tilde{a}), S(\tilde{\pi}) \) be the score values of \( a \) and \( \pi \) respectively. Based on the score value of PFN in (3) and the above inequality, we can imply that \( S(\tilde{a}) \geq S(\tilde{\pi}) \), and then we discuss the following cases:

i. If \( S(\tilde{a}) > S(\tilde{\pi}) \), then we can get \( \text{PFHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) > \text{PFHM}^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k) \).

ii. If \( S(\tilde{a}) = S(\tilde{\pi}) \), then

\[
\frac{1}{2} \left( 1 + \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \right)^{\frac{1}{k}} \right) \leq \left( \prod_{j=1}^{x} \left( 1 - (v_j)^2 \right) \right)^{\frac{1}{k}} \leq \left( \prod_{j=1}^{x} \left( 1 - (\pi \theta_j)^2 \right) \right)^{\frac{1}{k}}
\]

Since \( \mu_i \geq \mu_0, v_0 \geq v_j \geq 0 \), we can deduce that

\[
\sqrt{1 - \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}}} = \sqrt{1 - \left( \prod_{1 \leq i_1 \cdots < i_k \leq k} \left( 1 - \left( \prod_{i=1}^{x} \mu_0 \right)^{\frac{1}{2}} \right) \right)^{\frac{i}{k}}}
\]
And
\[
\left( \prod_{1 \leq i_1 < \cdots < i_t \leq k} \sqrt{1 - \left( \prod_{j=1}^{x} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}}} \right)_{\mathcal{T}\frac{t}{k}} = \left( \prod_{1 \leq i_1 < \cdots < i_t \leq k} \sqrt{1 - \left( \prod_{j=1}^{x} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}}} \right)_{\mathcal{T}\frac{t}{k}}
\]

Therefore, it follows that
\[
H(\bar{a}) = \left( \left( \prod_{1 \leq i_1 < \cdots < i_t \leq k} \left( 1 - \left( \prod_{j=1}^{\pi} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}} \right) \right)^{-\frac{1}{2}} \right)^{2} = \left( \left( \prod_{1 \leq i_1 < \cdots < i_t \leq k} \left( 1 - \left( \prod_{j=1}^{\pi} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}} \right) \right)^{-\frac{1}{2}} \right)^{2} = H(\bar{a})
\]

The PFHM\(^{(x)}\)\((\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \text{PFHM}\(^{(x)}\)(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k). □

**Property 3.** (Boundedness). Let \(\bar{a}_i = (\mu_{i_j}, v_{i_j}), \bar{a}^+ = (\mu_{\max_i}, v_{\max_i})\) (\(i = 1, 2, \cdots, k\)) be a set of PFNs, and \(\bar{a}^- = (\mu_{\min_i}, v_{\min_i})\) then
\[
\bar{a}_i^- < \text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) < \bar{a}_i^+
\]

**Proof.** Based on Properties 1 and 2, we have
\[
\text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) \geq \text{PFHM}\(^{(x)}\)(\bar{a}^-, \bar{a}^-, \cdots, \bar{a}^-) = \bar{a}^-,
\]
\[
\text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) \leq \text{PFHM}\(^{(x)}\)(\bar{a}^+, \bar{a}^+, \cdots, \bar{a}^+) = \bar{a}^+.
\]

Then we have \(\bar{a}^- \leq \text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) \leq \bar{a}^+ □\)

**Property 4.** (Commutativity). Let \(\bar{a}_i = (\mu_{i_j}, v_{i_j}), \bar{\pi}_i = (\mu_{\theta_j}, v_{\theta_j})\) (\(i = 1, 2, \cdots, k\)) be two sets of PFNs. Suppose \((\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_k)\) is any permutation of \((\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k)\), then
\[
\text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \text{PFHM}\(^{(x)}\)(\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_k)
\]

**Proof.** Because \((\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_k)\) is any permutation of \((\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k)\), then
\[
\frac{\prod_{1 \leq i_1 < \cdots < i_t \leq k} \left( \prod_{j=1}^{\pi} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}}}{\mathcal{T}\frac{t}{k}} = \frac{\prod_{1 \leq i_1 < \cdots < i_t \leq k} \left( \prod_{j=1}^{\pi} \left( 1 - (v_{i_j})^2 \right) \right)^{\frac{1}{2}}}{\mathcal{T}\frac{t}{k}}
\]
Thus, \(\text{PFHM}\(^{(x)}\)(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \text{PFHM}\(^{(x)}\)(\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_k). □\)

Next, we will discuss some particular cases of PFHM operator when \(x\) takes different values.
Case 1: When \( x = 1 \), then PFHM operator will become arithmetic average operator of PFNs.

\[
\text{PFHM}^{(1)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( 1 - \left( \prod_{1 \leq i < \cdot < i \leq k} \left( 1 - \mu_i^2 \right) \right)^{\frac{1}{k}}, \prod_{1 \leq i < \cdot < i \leq k} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{k}}
\]

\begin{equation}
\left( \prod_{1 \leq i < \cdot < i \leq k} \left( 1 - \mu_i^2 \right) \right)^{\frac{1}{k}}, \prod_{1 \leq i < \cdot < i \leq k} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{k}} = \frac{1}{k} \oplus \tilde{a}_i
\end{equation}

Case 2: When \( x = k \), then PFHM operator will become arithmetic average operator of PFNs.

\[
\text{PFHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( 1 - \left( \prod_{1 \leq i < \cdot < i \leq k} \left( 1 - \mu_i^2 \right) \right)^{\frac{1}{k}}, \prod_{1 \leq i < \cdot < i \leq k} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{k}}
\]

\begin{equation}
\left( \prod_{1 \leq i < \cdot < i \leq k} \left( 1 - \mu_i^2 \right) \right)^{\frac{1}{k}}, \prod_{1 \leq i < \cdot < i \leq k} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{k}} = \frac{k}{\sum_{i=1}^{k} \mu_i}
\end{equation}

Example 3. Let \( \tilde{a}_1 = (0.5, 0.2), \tilde{a}_2 = (0.6, 0.3), \tilde{a}_3 = (0.4, 0.1), \tilde{a}_4 = (0.7, 0.3) \) be four PFNs. Then we use the proposed PFHM operator to aggregate four PFNs. (suppose \( x = 2 \))

\[
\tilde{a} = \text{PFHM}^{(2)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k)
\]

\[
\left( 1 - \left( \prod_{1 \leq i < \cdot < i \leq 4} \left( 1 - \left( \prod_{j=1}^{x} \mu_i \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{x}}, \prod_{1 \leq i < \cdot < i \leq 4} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{x}}
\]

\[
\left( 1 - \left( \prod_{1 \leq i < \cdot < i \leq 4} \left( 1 - \left( \prod_{j=1}^{x} \mu_i \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{x}}, \prod_{1 \leq i < \cdot < i \leq 4} \sqrt{1 - \left( v_i \right)^2} \right)^{\frac{1}{x}} = \frac{1 \times (1 - 0.5 \times 0.6) \times (1 - 0.5 \times 0.4) \times (1 - 0.5 \times 0.7)}{(1 - 0.6 \times 0.4) \times (1 - 0.6 \times 0.7) \times (1 - 0.4 \times 0.7)^{\frac{1}{x}}}
\]

At last, we get \( \text{PFHM}^{(2)}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (0.5497, 0.2325) \).

3.2. The WPFHM Operator

The weights of attributes play an important role in practical decision making, and they can influence the decision result. Therefore, it is necessary to consider attribute weights in aggregating information. It is obvious that the PFHM operator fails to consider the problem of attribute weights. In order to overcome this defect, we propose the WPFHM operator.
Definition 9. Let \( \tilde{a}_i = (\mu_i, v_i)(i = 1, 2, \cdots, k) \) be a group of PFNs, \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^T \) be the weight vector for \( \tilde{a}_i(i = 1, 2, \cdots, k) \), which satisfies \( \omega_i \in [0,1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \), then we can define WPFHM operator as follows:

\[
\text{WPFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left\{ \begin{array}{ll}
\frac{1}{\omega} \sum_{i=1}^{x} \left( 1 - \left( \prod_{j=1}^{i}(1 - (v_j)^2) \right)^{\frac{1}{2}} \right) (1 \leq x < k) \\
\frac{1}{\omega} \sum_{i=1}^{k} \left( 1 - \left( \prod_{j=1}^{i}(1 - (v_j)^2) \right)^{\frac{1}{2}} \right) (x = k)
\end{array} \right.
\]

Theorem 2. Let \( \tilde{a}_i = (\mu_i, v_i)(i = 1, 2, \cdots, k) \) be a group of PFNs, and their weight vector meet \( \omega_i \in [0,1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \) then the result from Definition 9 is still a PFN, and have.

\[
\text{WPFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left\{ \begin{array}{ll}
\frac{1}{\omega} \sum_{i=1}^{x} \left( 1 - \left( \prod_{j=1}^{i}(1 - (v_j)^2) \right)^{\frac{1}{2}} \right) (1 \leq x < k) \\
\frac{1}{\omega} \sum_{i=1}^{k} \left( 1 - \left( \prod_{j=1}^{i}(1 - (v_j)^2) \right)^{\frac{1}{2}} \right) (x = k)
\end{array} \right.
\]

Or

\[
\text{WPFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left( \prod_{i=1}^{k} \left( 1 - \frac{\omega_i}{\sum_{i=1}^{k} \omega_i} \right) \right)^{\frac{1}{2}} \left( 1 - \left( \prod_{j=1}^{i}(1 - (v_j)^2) \right)^{\frac{1}{2}} \right) (x = k)
\]

Proof.

1. First of all, we prove that (19) and (20) are kept. For the first case, when \( 1 \leq x < k \), according to the operational laws of PFNs, we get

\[
\left( \prod_{j=1}^{i} \tilde{a}_i \right)^{\frac{1}{2}} = \left( \prod_{j=1}^{i} \mu_j \right)^{\frac{1}{2}} \left( \prod_{j=1}^{i} (1 - (v_j)^2) \right)^{\frac{1}{2}}
\]

Thereafter,

\[
\left( 1 - \sum_{j=1}^{i} \omega_j \right) \left( \prod_{j=1}^{i} \tilde{a}_i \right)^{\frac{1}{2}} = \left( 1 - \left( \prod_{j=1}^{i} \mu_j \right)^{\frac{1}{2}} \left( \prod_{j=1}^{i} (1 - (v_j)^2) \right)^{\frac{1}{2}} \right)
\]

(21)
Moreover,

\[
\left[1 - \bigtimes_{1 \leq i_1 < \cdots < i_k \leq k} \frac{1}{1 - \sum_{j=1}^{k} \omega_j} \left( \frac{1}{\prod_{j \in [1,k]} \mu_j} \right) \right]^2
= \left[1 - \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{k} \mu_j \right)^{\frac{1}{x_j}} \right) \right]^{2 \left(1 - \frac{1}{x_j} \sum_{j=1}^{k} \omega_j \right)}
\]  

Therefore,

\[
\left[1 - \bigtimes_{1 \leq i_1 < \cdots < i_k \leq k} \frac{1}{1 - \sum_{j=1}^{k} \omega_j} \left( \frac{1}{\prod_{j \in [1,k]} \mu_j} \right) \right]^\frac{1}{x_j - 1}
= \left[1 - \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{k} \mu_j \right)^{\frac{1}{x_j}} \right) \right]^{\frac{1}{x_j - 1} \left(1 - \frac{1}{x_j} \sum_{j=1}^{k} \omega_j \right)}
\]  

For the second case, when \((x = k)\), we get

\[
\tilde{a}_i^{\frac{1 - \omega_j}{x_j - 1}} = \left( \frac{1}{x_j} \right)^{\frac{1 - \omega_j}{x_j - 1}} \sqrt{1 - \left( 1 - \nu_i \right)^{\frac{1 - \omega_j}{x_j - 1}}}
\]  

Then,

\[
\prod_{i=1}^{k} \tilde{a}_i^{\frac{1 - \omega_j}{x_j - 1}} = \left( \frac{1}{x_j} \right)^{\frac{1 - \omega_j}{x_j - 1}} \sqrt{1 - \prod_{i=1}^{k} \left( 1 - \nu_i \right)^{\frac{1 - \omega_j}{x_j - 1}}}
\]  

(2) Next, we prove the (19) and (20) are PFNs. For the first case, when \(1 \leq x < k\),

Let

\[
p = \sqrt{1 - \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{k} \mu_j \right)^{\frac{1}{x_j}} \right) \right)^{2 \left(1 - \frac{1}{x_j} \sum_{j=1}^{k} \omega_j \right)}}
\]  

\[
q = \left[1 - \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \nu_i \right)^{\frac{1}{x_j}} \right) \right)^{\frac{1}{x_j - 1} \left(1 - \frac{1}{x_j} \sum_{j=1}^{k} \omega_j \right)} \right]^{\frac{1}{x_j - 1}}
\]  

Then we need prove the following two conditions. (i) \(0 \leq p \leq 1, 0 \leq q \leq 1\). (ii) \(0 \leq p^2 + q^2 \leq 1\).
i. Since $p \in [0, 1]$, we can get

$$\left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \in [0, 1] \Rightarrow \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \in [0, 1] \Rightarrow 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \in [0, 1]$$

$$\Rightarrow \prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \in [0, 1] \Rightarrow \prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \in [0, 1]$$

$$\Rightarrow 1 - \left( \prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \right) \in [0, 1]$$

Therefore, $0 \leq p \leq 1$.

Similarly, we can get

$$\prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \in [0, 1]$$

Therefore, $0 \leq q \leq 1$.

ii. Since $0 \leq p^2 + q^2 \leq 1$, we can get the following inequality:

$$\left\{ \left( 1 - \frac{1}{\prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) } \right)^{\frac{1}{i}} \right\}^{2} \leq 1 - \frac{1}{\prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) } \leq 1$$

□

For the second case, when $x = k$, we can easily prove that it is kept. So the aggregation result produced by Definition 8 is still a PFN. Next, we shall deduce some desirable properties of WPFHM operator.

**Property 5.** (Idempotency). If $\tilde{a}_i (i = 1, 2, \cdots, k)$ are equal, i.e., $\tilde{a}_i = \tilde{a} = (\mu, \nu)$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$ then

$$\text{WPFHM}_{12}^{(k)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \tilde{a}$$

(27)

**Proof.** Since $\tilde{a}_i = \tilde{a} = (\mu, \nu)$, based on Theorem 2, we get

(1) For the first case, when $1 \leq x < k$. 

$$\text{WPFHM}_{12}^{(k)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)$$

$$= \left( \left\{ \left( \prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \right)^{v} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{i}} \right) \right)^{u} \right\} \frac{1}{v+1} \right)^{\frac{1}{v+1}}$$
For the second case, when $x = k$, symmetric.

If $j \prod_{i=1}^{j} \prod_{i=1}^{j-1} \prod_{i=1}^{j-1} \prod_{i=1}^{j-1}$

Since $\sum_{i=1}^{k} \omega_i = 1$, we get $WPFH_{x \omega}^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k) = \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{\omega}}, \left( \prod_{i=1}^{k} \left( 1 - (1 - (v)^2) \right)^{\frac{1}{\omega}} \right) = \left( \mu, v \right) = \bar{a}$, which proves the idempotency property of the WPFHM operator. □

(2) For the second case, when $x = k$,

$WPFH_{x \omega}^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k) = \left( \prod_{i=1}^{k} \mu_i \right)^{\frac{1}{\omega}}, \left( \prod_{i=1}^{k} \left( 1 - (1 - (v)^2) \right)^{\frac{1}{\omega}} \right) = \left( \mu, v \right) = \bar{a}$

$WPFH_{x \omega}^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k) \geq WPFH_{x \omega}^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k)$

**Proof.** Since $x \geq 1, \mu_i \geq \mu_\theta, v_\theta \geq v_j \geq 0$, then

(w \theta \mu_\theta) \geq 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{\omega}} \Rightarrow \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{\omega}} \geq \left( \prod_{i=1}^{x} \mu_\theta \right)^{\frac{1}{\omega}} \Rightarrow 1 - \left( \prod_{i=1}^{x} \mu_\theta \right)^{\frac{1}{\omega}} \leq 1 - \left( \prod_{i=1}^{x} \mu_i \right)^{\frac{1}{\omega}}$
If

\[ \prod_{1 \leq i < \cdots < n \leq k} \left( 1 - \left( \prod_{j=1}^{x} \mu_{i \mid j} \right)^{\frac{1}{x}} \right)^{2} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right) \]

\[ \leq \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} \mu_{i \mid j} \right)^{\frac{1}{x}} \right)^{2} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} \]

Similarly, we have

\[ 1 - (v_{j})^{2} \geq 1 - (v_{\theta})^{2} \Rightarrow \left( \prod_{j=1}^{x} (1 - (v_{j})^{2}) \right)^{\frac{1}{x}} \geq \left( \prod_{j=1}^{x} (1 - (v_{\theta})^{2}) \right)^{\frac{1}{x}} \]

\[ \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{j})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \leq \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{\theta})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \]

Let \( a = \text{WPFHM}(\bar{a}_{1}, \bar{a}_{2}, \cdots, \bar{a}_{k}) \), \( \pi = \text{WPFHM}(\bar{\pi}_{1}, \bar{\pi}_{2}, \cdots, \bar{\pi}_{k}) \) and \( S(a), S(\pi) \) be the score values of \( a \) and \( \pi \) respectively. Based on the score value of PFN in (3) and the above inequality, we can imply that \( S(a) \geq S(\pi) \), and then we discuss the following cases:

1. If \( S(a) > S(\pi) \), then we can get \( \text{PFHM}(\bar{a}_{1}, \bar{a}_{2}, \cdots, \bar{a}_{k}) > \text{PFHM}(\bar{\pi}_{1}, \bar{\pi}_{2}, \cdots, \bar{\pi}_{k}) \).
2. If \( S(a) = S(\pi) \), then

\[ \frac{1}{2} \cdot \left( \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{j})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \right) - \left( \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{\theta})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \right) \]

Since \( \mu_{ij} \geq \mu_{\theta j} \geq 0, v_{\theta j} \geq v_{j} \geq 0 \), and based on the Equations (3) and (4), we can deduce that

\[ \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{j})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } = \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{\theta})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \]

And

\[ \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{j})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } = \prod_{1 \leq i < \cdots < k \leq \varphi} \left( 1 - \left( \prod_{j=1}^{x} (1 - (v_{\theta})^{2}) \right)^{\frac{1}{x}} \right)^{ \frac{1}{x} \left( 1 - \frac{\sum_{j=1}^{x} \omega_{j}}{x} \right)^{\frac{1}{x}} } \]

Therefore, it follows that \( H(\bar{a}) = H(\bar{\pi}) \), the \( \text{WPFHM}(\bar{a}_{1}, \bar{a}_{2}, \cdots, \bar{a}_{k}) = \text{WPFHM}(\bar{\pi}_{1}, \bar{\pi}_{2}, \cdots, \bar{\pi}_{k}) \).
When $x = k$, we can prove it in a similar way.

**Property 7.** (Boundedness). Let $\tilde{a}_i = (\mu_{i1}, v_{i1}), \tilde{a}^+ = (\mu_{\text{max}_i}, v_{\text{max}_i}) (i = 1, 2, \cdots, k)$ be a set of PFNs, and $\tilde{a}^- = (\mu_{\text{min}_i}, v_{\text{min}_i})$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$ then

$$\tilde{a}^- \leq \text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \leq \tilde{a}^+ \quad (29)$$

**Proof.** Based on Properties 5 and 6, we have

$$\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \geq \text{WPWFHM}^{(x)}_{\omega}(\tilde{a}^-, \tilde{a}^-, \cdots, \tilde{a}^-) = \tilde{a}^-,$$

$$\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \leq \text{WPWFHM}^{(x)}_{\omega}(\tilde{a}^+, \tilde{a}^+, \cdots, \tilde{a}^+) = \tilde{a}^+.\Box$$

**Property 8.** (Commutativity). Let $\tilde{a}_i = (\mu_{i1}, v_{i1}), \tilde{\pi}_i = (\mu_{i\theta_j}, v_{\theta_j}) (i = 1, 2, \cdots, k)$ be two sets of PFNs. Suppose $(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$, the $\tilde{a}$ and $\tilde{\pi}$ are equal, then we have

$$\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \text{WPWFHM}^{(x)}_{\omega}(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k) \quad (30)$$

**Proof.** Because $(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)$, then

$$\frac{\oplus_{1 \leq i < j \leq k} \left( 1 - \sum_{i=1}^{k} \omega_i \right)^{\frac{1}{\text{C}_{k-1}^2}} \left( \frac{x}{\text{C}_{k-1}^2} \right)^{\frac{1}{\text{C}_{k-1}^2}}}{1 \leq i \leq \prod_{1 \leq i \leq k} \left( 1 - (\mu_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right)^{\frac{1}{\text{C}_{k-1}^2}} (x = k)}$$

Thus, $\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \text{WPWFHM}^{(x)}_{\omega}(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k)$. Next, we will discuss some particular cases of WPWFHM operator for different value $x.\Box$

**Case 1:** When $x = 1$, the WPWFHM will reduce to the following form:

$$\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \frac{\oplus_{1 \leq i < j \leq k} \left( 1 - \sum_{i=1}^{k} \omega_i \right)^{\frac{1}{\text{C}_{k-1}^2}} \left( \frac{x}{\text{C}_{k-1}^2} \right)^{\frac{1}{\text{C}_{k-1}^2}}}{1 \leq i \leq \prod_{1 \leq i \leq k} \left( 1 - (\mu_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right)^{\frac{1}{\text{C}_{k-1}^2}} (x = k)}$$

**Case 2:** When $x = k$, the proposed WPWFHM operator will reduce to the following form:

$$\text{WPWFHM}^{(x)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \frac{\prod_{1 \leq i \leq k} \left( 1 - (\mu_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right)^{\frac{1}{\text{C}_{k-1}^2}}}{1 \leq i \leq \prod_{1 \leq i \leq k} \left( 1 - (\mu_{i1})^2 \right) \left( 1 - (v_{i1})^2 \right)^{\frac{1}{\text{C}_{k-1}^2}} (x = k)}$$
Example 4. Let $\tilde{a}_1 = (0.6, 0.4), \tilde{a}_2 = (0.7, 0.3), \tilde{a}_3 = (0.5, 0.1), \tilde{a}_4 = (0.4, 0.3)$ be four PFNs. The weighting vector of attributes be $\omega = \{0.1, 0.3, 0.4, 0.2\}$. Then we use the proposed WPFHM operator to aggregate four PFNs. (suppose $x = 2$)

\[
\tilde{a} = \text{WPFHM}_\omega^{(2)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_4) = \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

\[
= \left(1 - \prod_{1 \leq k < j \leq 4} \left(1 - \left(\prod_{j=1}^{k} \mu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \nu_j \right)^{\frac{1}{2}} \left(1 - \sum_{j=1}^{k} \omega_j \right)^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \]

At last, we get WPFHM$_\omega^{(2)}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (0.7015, 0.3114)$.

3.3. The PFDMH Operator

Wu et al. [59] proposed the DHM operator.

Definition 10. The DHM operator is defined [59]:

\[
\text{DHM}^{(x)}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \prod_{1 \leq i_1 < \ldots < i_x \leq n} \left(\frac{\sum_{j=1}^{x} \varphi_{i_j}}{x}\right)^{\frac{1}{x}} \tag{33}
\]

where $x$ is a parameter and $x = 1, 2, \ldots, k$, $i_1, i_2, \ldots, i_x$ are $x$ integer values taken from the set $\{1, 2, \ldots, k\}$ of $k$ integer values, $C^x_n$ denotes the binomial coefficient and $C^x_k = \frac{k^!}{x!(k-x)^!}$.

In this section, we propose the Pythagorean fuzzy DHM (PFDMH) operator.

Definition 11. Let $\tilde{a}_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, k)$ be a set of PFNs, then we define PFDMH operator as follows:

\[
\text{PFDMH}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left(1 \otimes \sum_{1 \leq i_1 < \ldots < i_x \leq k} \left(\frac{\sum_{j=1}^{x} \tilde{a}_{i_j}}{x}\right)^{\frac{1}{x}} \right) \tag{34}
\]

where $x$ is a parameter and $x = 1, 2, \ldots, k$, $i_1, i_2, \ldots, i_x$ are $x$ integer values taken from the set $\{1, 2, \ldots, k\}$ of $k$ integer values, $C^x_n$ denotes the binomial coefficient and $C^x_k = \frac{k^!}{x!(k-x)^!}$.
Theorem 3. Let $\tilde{a}_i = (\mu_i, v_i)(i = 1, 2, \cdots, k)$ be a collection of the PFNs, then the aggregate result of definition 10 is still a PFNs, and have

$$\text{PFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left( \prod_{1 \leq h < \cdots < i, k \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right] \right) \left[ 1 - \left( \prod_{i=1}^{k} v_i \right)^{\frac{1}{\nu}} \right]$$

(35)

Proof.

(1) First of all, we prove (35) is kept.

$$\frac{x}{\nu} \tilde{a}_i = \left( \prod_{j=1}^{x} \left( 1 - \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right) \left( \prod_{i=1}^{x} v_i \right)^{\frac{1}{\nu}}$$

(36)

Then,

$$\frac{x}{\nu} \tilde{a}_i = \left( \prod_{j=1}^{x} \left( 1 - \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right) \left( \prod_{i=1}^{x} v_i \right)^{\frac{1}{\nu}}$$

(37)

Thereafter,

$$\left( \prod_{1 \leq h < \cdots < i, k \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right] \right) \left[ 1 - \left( \prod_{i=1}^{k} v_i \right)^{\frac{1}{\nu}} \right]$$

(38)

Furthermore,

$$\text{PFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \left( \prod_{1 \leq h < \cdots < i, k \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right] \right) \left[ 1 - \left( \prod_{i=1}^{k} v_i \right)^{\frac{1}{\nu}} \right]$$

(39)

(2) Next, we prove (34) is a PFN. Let

$$p = \left( \prod_{1 \leq h < \cdots < i, k \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right] \right) \left[ 1 - \left( \prod_{i=1}^{k} v_i \right)^{\frac{1}{\nu}} \right]$$

$$q = \left[ 1 - \left( \prod_{1 \leq h < \cdots < i, k \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_j)^2) \right)^{\frac{x}{\nu}} \right] \right) \right] \left[ 1 - \left( \prod_{i=1}^{k} v_i \right)^{\frac{1}{\nu}} \right]$$

Then we need to prove that it satisfies the following two conditions. (i) $0 \leq p \leq 1, 0 \leq q \leq 1$; (ii) $0 \leq p^2 + q^2 \leq 1$. 
i. Since $\mu_{ij} \in [0, 1]$, we can get

$$(\mu_{ij})^2 \in [0, 1] \Rightarrow 1 - (\mu_{ij})^2 \in [0, 1] \Rightarrow \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \in [0, 1]$$

$$(\mu_{ij})^2 \in [0, 1] \Rightarrow 1 - (\mu_{ij})^2 \in [0, 1] \Rightarrow \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \in [0, 1]$$

$$\Rightarrow 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \in [0, 1] \Rightarrow \prod_{1 \leq i < \cdots < q \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right] \in [0, 1]$$

$$\Rightarrow \left( \prod_{1 \leq i < \cdots < q \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right] \right)^{\frac{1}{q}} \in [0, 1], \text{ therefore, } 0 \leq q \leq 1.$$}

ii. Obviously, $0 \leq p^2 + q^2 \leq 1$, then

$$\left( \prod_{1 \leq i < \cdots < q \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right] \right)^{\frac{1}{q}} + 1 - \left( \prod_{1 \leq i < \cdots < q \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right] \right)^{\frac{1}{q}} = 1$$

We get $0 \leq p^2 + q^2 \leq 1$. □

So the aggregated result of Definition 10 is still PFN. Next we will talk about some properties of PFDHM operator.

**Property 9. (Idempotency).** If $\tilde{a}_i(1, 2, \cdots, k)$ and $\tilde{a}$ are PFNs, and $\tilde{a}_i = \tilde{a} = (\mu_i, v_i)$ for all $i = 1, 2, \cdots, k$, then we get

$$\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \tilde{a}$$

(40)  

**Proof.**

$$\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)$$

$$= \left( \prod_{1 \leq i < \cdots < q \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right] \right)^{\frac{1}{q}} + \left[ 1 - \left( \prod_{1 \leq i < \cdots < q \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - (\mu_{ij})^2) \right)^{\frac{1}{4}} \right) \right) \right]^{\frac{1}{q}}$$
1. If \( S \) values of \( \text{Symmetry} \) imply that \( S \) Property 10. (Monotonicity). Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \), \( v = (v_1, v_2, \ldots, v_k) \) be two sets of PFNs. If \( \mu_i \geq \mu_j, v_i \leq v_j \) for all \( j \), then

\[
PFDHM^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \geq PFDHM^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)
\]  

(41)

**Proof.** Since \( x \geq 1, \mu_i \geq \mu_j \geq 0, v_0 \geq v_j \geq 0 \), then

\[
\left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \leq \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \Rightarrow 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \geq 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}}
\]

\[
\left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \leq \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \Rightarrow 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \geq 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}}
\]

\[
\Rightarrow \left( \prod_{1 \leq i_1 < \ldots < i_s \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \right] \right)^{\frac{1}{x}} \geq \left( \prod_{1 \leq i_1 < \ldots < i_s \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}} \right] \right)^{\frac{1}{x}}
\]

Similarly, we have

\[
\left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right)^{2} \leq \left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right)^{2} \Rightarrow 1 - \left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right)^{2} \geq 1 - \left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right)^{2}
\]

\[
\Rightarrow \left( \prod_{1 \leq i_1 < \ldots < i_s \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right) \right] \right)^{\frac{1}{x}} \geq \left( \prod_{1 \leq i_1 < \ldots < i_s \leq k} \left[ 1 - \left( \prod_{j=1}^{x} (v_j)^{\frac{1}{x}} \right) \right] \right)^{\frac{1}{x}}
\]

Let \( \tilde{a} = PFDHM^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k), \tilde{\pi} = PFDHM^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k) \) and \( S(\tilde{a}), S(\tilde{\pi}) \) be the score values of \( \tilde{a} \) and \( \tilde{\pi} \) respectively. Based on the score value of PFN in (3) and the above inequality, we can imply that \( S(\tilde{a}) \geq S(\tilde{\pi}) \), and then we discuss the following cases:

1. If \( S(\tilde{a}) > S(\tilde{\pi}) \), then we can get \( PFDHM^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) > PFDHM^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k) \)
2. If $S(\tilde{a}) = S(\tilde{\pi})$, then
\[
\frac{1}{2} \left( 1 + \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2} \right) \right)^2 - \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2} \right)^2 = \frac{1}{2} \left( 1 + \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\nu_{ij})^2) \right)^2} \right) \right)^2 - \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\nu_{ij})^2) \right)^2} \right)^2
\]

Since $\mu_{ij} \geq \mu_{j} \geq 0$, $\nu_{ij} \geq v_{ij} \geq 0$, and based on the Equations (3) and (4), we can deduce that
\[
\prod_{1 \leq i < \cdots < i \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2 \right]^{\frac{1}{2}} = \prod_{1 \leq i < \cdots < i \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2 \right]^{\frac{1}{2}}
\]

And
\[
\prod_{1 \leq i < \cdots < i \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2 \right]^{\frac{1}{2}} = \prod_{1 \leq i < \cdots < i \leq k} \left[ 1 - \left( \prod_{j=1}^{i} (1 - (\nu_{ij})^2) \right)^2 \right]^{\frac{1}{2}}
\]

Therefore, it follows that
\[
H(\tilde{a}) = \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\mu_{ij})^2) \right)^2} \right)^2 + \left( \prod_{1 \leq i < \cdots < i \leq k} \sqrt{1 - \left( \prod_{j=1}^{i} (1 - (\nu_{ij})^2) \right)^2} \right)^2 = H(\tilde{\pi})
\]

The \(\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)\) is PFDHM\(^{(x)}\) \((\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k)\)

**Property 11.** (Boundedness). Let $\tilde{a}_i = (\mu_{ij}, v_{ij}), \tilde{a}_i^+ = (\mu_{\max_i}, v_{\max_i}) (i = 1, 2, \cdots, k)$ be a set of PFNs, and $\tilde{a}^- = (\mu_{\min_i}, v_{\min_i})$ then
\[
\tilde{a}_i^- \leq \text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) < \tilde{a}_i^+ \tag{42}
\]

**Proof.** Based on Properties 9 and 10, we have
\[
\begin{align*}
\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) & \geq \text{PFDHM}^{(x)}(\tilde{a}^-, \tilde{a}^-, \cdots, \tilde{a}^-) = \tilde{a}^-, \\
\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) & \leq \text{PFDHM}^{(x)}(\tilde{a}^+, \tilde{a}^+, \cdots, \tilde{a}^+) = \tilde{a}^+,
\end{align*}
\]

Then we have $\tilde{a}^- \leq \text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \leq \tilde{a}^+$.

**Property 12.** (Commutativity). Let $\tilde{a}_i = (\mu_{ij}, v_{ij}), \tilde{\pi}_i = (\mu_{\pi_i}, v_{\pi_i}) (i = 1, 2, \cdots, k)$ be two sets of PFNs. Suppose $(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k)$, then
\[
\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \text{PFDHM}^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_k) \tag{43}
\]
**Proof.** Because \((\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)\) is any permutation of \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k)\), then

\[
\left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( \frac{\tilde{x} \oplus \tilde{a}_{i_j}}{x} \right) \right)^{\frac{1}{x}} = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( \frac{\tilde{x} \oplus \tilde{\pi}_{i_j}}{x} \right) \right)^{\frac{1}{x}}
\]

Thus, \(\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \text{PFDHM}^{(x)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)\).

Next, we discuss some particular cases of PFDHM operator.

**Case 1:** When \(x = 1\), then PFDHM operator will become arithmetic average operator of PFNs.

\[
\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - \mu_x^{2}) \right) \right)^{\frac{1}{x}} \right)
\]

**Case 2:** When \(x = k\), then PFDHM operator will become arithmetic average operator of PFNs.

\[
\text{PFDHM}^{(x)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( \frac{\tilde{x} \oplus \tilde{a}_{i_j}}{x} \right) \right)^{\frac{1}{x}}
\]

**Example 5.** Let \(\tilde{a}_1 = (0.6, 0.2), \tilde{a}_2 = (0.5, 0.3), \tilde{a}_3 = (0.7, 0.1), \tilde{a}_4 = (0.8, 0.2)\) be four PFNs. Then we use the PFDHM operator to fuse four PFNs. (suppose \(x = 2\))

\[
\tilde{a} = \text{PFDHM}^{(2)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = (\mu, v)
\]

\[
= \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - \mu_x^{2}) \right) \right)^{\frac{1}{x}} \right)^{\frac{1}{x}}
\]

\[
= \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( 1 - \left( \prod_{j=1}^{x} (1 - \mu_x^{2}) \right) \right)^{\frac{1}{x}} \right)^{\frac{1}{x}}
\]
Theorem 4. Let \( \text{Symmetry} \). Let  

\[  \text{WPFDHM} = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) = (0.6627, 0.1962). \]

3.4. The WPFDHM Operator  

The weights of attributes play an important role in practical decision making, and they can influence the decision result. Therefore, it is necessary to consider attribute weights in aggregating information. It is obvious that the PFDHM operator fails to consider the problem of attribute weights. In order to overcome this defect, we propose the WPFDHM operator.

**Definition 12.** Let \( \bar{a}_i = (\mu_i, v_i) \) \( i = 1, 2, \cdots, k \) be a group of PFNs, \( \omega = (\omega_1, \omega_2, \cdots, \omega_k) \) be the weight vector for \( \bar{a}_i \) \( i = 1, 2, \cdots, k \), which satisfies \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \), then we can define WPFDHM operator as follows:

\[
\text{WPFDHM}^{(x)}_{\omega}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \begin{cases} 
\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \sum_{j=1}^{x} \omega_{i_j} \right) \left( \frac{\bar{a}_{i_1} \cdots \bar{a}_{i_k}}{\prod_{j=1}^{x} \omega_j} \right) \left( \frac{1}{x} \right), & (1 \leq x < k) \\
\prod_{i=1}^{x} \bar{a}_i \prod_{i=1}^{1-x} \bar{a}_i \left( \frac{1}{x} \right) \prod_{i=1}^{1-x} \left( \frac{1}{x} \right), & (x = k) 
\end{cases} \tag{44}
\]  

**Theorem 4.** Let \( \bar{a}_j = (\mu_j, v_j) \) \( i = 1, 2, \cdots, k \) be a group of PFNs, and their weight vector meet \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \) then the result from Definition 11 is still a PFN, and have

\[
\text{WPFDHM}^{(x)}_{\omega}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \begin{cases} 
\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( 1 - \sum_{j=1}^{x} \omega_{i_j} \right) \left( \frac{\bar{a}_{i_1} \cdots \bar{a}_{i_k}}{\prod_{j=1}^{x} \omega_j} \right) \left( \frac{1}{x} \right), & (1 \leq x < k) \\
\prod_{i=1}^{x} \bar{a}_i \prod_{i=1}^{1-x} \bar{a}_i \left( \frac{1}{x} \right) \prod_{i=1}^{1-x} \left( \frac{1}{x} \right), & (x = k) \tag{45}
\end{cases}
\]

Or

\[
\text{WPFDHM}^{(x)}_{\omega}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k) = \prod_{i=1}^{x} \bar{a}_i \prod_{i=1}^{1-x} \bar{a}_i \left( \frac{1}{x} \right) \prod_{i=1}^{1-x} \left( \frac{1}{x} \right), \quad (x = k) \tag{46}
\]

**Proof.**
(1) First of all, we prove that (45) and (46) are kept. For the first case, when \(1 \leq x < k\), we get
\[
\left( \frac{\tilde{a}_i}{x} \right)_{j=1}^{\frac{1}{x}} = \left( \sqrt{1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}}} \right)
\]
(47)

Then,
\[
\left( 1 - \sum_{j=1}^{x} \omega_j \right)^{\frac{1}{1-x}} = \left( \sqrt{1 - \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}}} \right)
\]
(48)

Thereafter,
\[
\left( \prod_{j=1}^{x} \left( 1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \left( \prod_{j=1}^{x} (1 - (\mu_j)^2) \right)^{\frac{1}{x}}
\]
(49)

Furthermore,
\[
\left( \prod_{j=1}^{x} \left( 1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \left( 1 - (\mu_j)^2 \right)^{\frac{1}{x}}
\]
(50)

For the second case, when \(x = k\), we get
\[
\tilde{a}_i^{\frac{1}{x}} = \left( \mu_i \right)^{\frac{1}{x}}, \sqrt{1 - (1 - (\mu_j)^2)^{\frac{1}{x}}}
\]
(51)

Then,
\[
\left( \prod_{j=1}^{x} \left( 1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \left( 1 - (\mu_j)^2 \right)^{\frac{1}{x}}
\]
(52)

(2) Next, we prove the (45) and (46) are PFNs. For the first case, when \(1 \leq x < k\)
\[
\rho = \left( \prod_{j=1}^{x} \left( 1 - (\mu_j)^2 \right)^{\frac{1}{x}} \right)^{\frac{1}{x}} \left( 1 - (\mu_j)^2 \right)^{\frac{1}{x}}
\]
\[
q = \sqrt{1 - \left( \prod_{j=1}^{x} \left( 1 - (\mu_j)^2 \right)^{\frac{1}{x}} \right)^{\frac{1}{x}}}
\]

Then we need prove the following two conditions. (i) \(0 \leq p \leq 1, 0 \leq q \leq 1\). (ii) \(0 \leq p^2 + q^2 \leq 1\).
i. Since $p \in [0, 1]$, we can get

\[
\prod_{j=1}^{x} \left(1 - (\mu_{ij})^{2}\right) \in [0, 1] \Rightarrow \left(\prod_{j=1}^{x} \left(1 - (\mu_{ij})^{2}\right)\right)^{\frac{1}{2}} \in [0, 1] \Rightarrow \sqrt{1 - \left(\prod_{j=1}^{x} \left(1 - (\mu_{ij})^{2}\right)\right)^{\frac{1}{2}}} \in [0, 1]
\]

\[
\Rightarrow \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^{x} \left(1 - (\mu_{ij})^{2}\right)\right)^{\frac{1}{2}}}\right)^{(1 - \sum_{j=1}^{x} \omega_{ij})} \in [0, 1]
\]

\[
\Rightarrow \left(\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(\sqrt{1 - \left(\prod_{j=1}^{x} \left(\mu_{ij}\right)^{2}\right)^{\frac{1}{2}}}\right)^{(1 - \sum_{j=1}^{x} \omega_{ij})}\right)^{1 - k-1} \in [0, 1]
\]

Therefore, $0 \leq p \leq 1$. Similarly, we can get

\[
\sqrt{1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(\mu_{ij}\right)^{2}\right)^{\frac{1}{2}}\right)^{2(1 - \sum_{j=1}^{x} \omega_{ij})}\right)^{\frac{1}{2}}} \in [0, 1].
\]

Therefore, $0 \leq q \leq 1$.

ii. Since $0 \leq p^2 + q^2 \leq 1$, we can get the following inequality:

\[
\left(\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(\mu_{ij}\right)^{2}\right)^{\frac{1}{2}}\right)^{(1 - \sum_{j=1}^{x} \omega_{ij})}\right)^{\frac{1}{2} + 1} \geq \left(\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(\mu_{ij}\right)^{2}\right)^{\frac{1}{2}}\right)^{(1 - \sum_{j=1}^{x} \omega_{ij})}\right)^{\frac{1}{2} + 1} + \left(\prod_{1 \leq i_1 < \cdots < i_k \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(\mu_{ij}\right)^{2}\right)^{\frac{1}{2}}\right)^{(1 - \sum_{j=1}^{x} \omega_{ij})}\right)^{\frac{1}{2} + 1} - 1
\]

For the second case, when $x = k$, we can easily prove that it is kept. So the aggregation result produced by Definition 8 is still a PFN. Next, we shall deduce some desirable properties of WPFHM operator.

**Property 13.** (Idempotency). If $\tilde{a}_i (i = 1, 2, \cdots, k)$ are equal, i.e., $\tilde{a}_i = \tilde{a} = (\mu, \nu)$ and weight vector meets $\omega_i \in [0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$ then

\[
\text{WPFHM}_\omega^{(x)}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \tilde{a}, \tag{53}
\]

**Proof.** Since $\tilde{a}_i = \tilde{a} = (\mu, \nu)$, based on Theorem 4, we get
(1) For the first case, when \(1 \leq x < k\).

\[
\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right)^{1-x \mu} \frac{1}{1-x \mu} \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu} \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu}
\]

Since \(\sum_{i=1}^{k} \omega_i = 1\), we can get

\[
\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu} \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu}
\]

(2) For the second case, when \(x = k\)

\[
\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu} \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu}
\]

Since \(\sum_{i=1}^{k} \omega_i = 1\), we can get

\[
\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu} \left( \prod_{i=1}^{k} \left( 1 - \left( \frac{1}{\omega_i} \right)^{1-x \mu_i} \right) \right) \frac{1}{1-x \mu}
\]

which proves the idempotency property of the WPFDHM operator. \(\square\)

**Property 14.** (Monotonicity). Let \(\tilde{a}_i = (\mu_i, v_i), \tilde{\pi}_i = (\mu_{i\theta}, v_{i\theta}) (i = 1, 2, \ldots, k)\) be two sets of PFNs. If \(\mu_i \geq \mu_{i\theta}, v_i \leq v_{i\theta}\) for all \(j\), and weight vector meets \(\omega_i \in [0, 1]\) and \(\sum_{i=1}^{k} \omega_i = 1\), the \(\tilde{a}\) and \(\tilde{\pi}\) are equal, then we have

\[
\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \geq \text{WPFDHM}^{(k)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)
\]
Proof. Since $x \geq 1, \mu_j \geq \mu_0 \geq 0, v_0 \geq v_j \geq 0$, then

$$1 - (\mu_j)^2 \leq 1 -(\mu_0)^2 \Rightarrow \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right) \leq \left( \prod_{j=1}^{x} \left(1 - (\mu_0)^2 \right) \right)^{\frac{1}{x}}$$

$$\Rightarrow \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right) \leq \left( \prod_{j=1}^{x} \left(1 - (\mu_0)^2 \right) \right)^{\frac{1}{x}}$$

Similarly, we have

$$\left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right) \leq \left( \prod_{j=1}^{x} \left(1 - (\mu_0)^2 \right) \right)^{\frac{1}{x}}$$

Let $a = \text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k), \pi = \text{WPFDHM}^{(k)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)$ and $S(a), S(\pi)$ be the score values of $a$ and $\pi$ respectively. Based on the score value of PFN in (3) and the above inequality, we can imply that $S(a) \geq S(\pi)$, and then we discuss the following cases:

1. If $S(a) > S(\pi)$, then we can get $\text{WPFDHM}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) > \text{WPFDHM}^{(k)}(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_k)$.

2. If $S(a) = S(\pi)$, then

$$= \frac{1}{2} \left[ \prod_{1 \leq i < \cdot < l_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \right) \right]^{-\frac{1}{x}} - \frac{1}{2} \left[ \prod_{1 \leq i < \cdot < l_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \right) \right]^{-\frac{1}{x}}$$

Since $\mu_j \geq \mu_0 \geq 0, v_0 \geq v_j \geq 0$, and based on the Equations (3) and (4), we can deduce that

$$\left( \prod_{1 \leq i < \cdot < l_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{x}} = \left( \prod_{1 \leq i < \cdot < l_k \leq k} \left( 1 - \left( \prod_{j=1}^{x} \left(1 - (\mu_j)^2 \right) \right)^{\frac{1}{x}} \right) \right)^{\frac{1}{x}}$$
And
\[
\left[ 1 - \prod_{1 \leq i < j \leq k} \left( 1 - \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_i} \right) \right]^{\frac{1}{1 - \omega_i}} = \left[ 1 - \prod_{1 \leq i < j \leq k} \left( 1 - \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_i} \right) \right]^{\frac{1}{1 - \omega_i}}
\]

Therefore, it follows that \( H(\tilde{a}) = H(\tilde{\alpha}) \), the WPFDHM\(^{(k)}\)(\(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k\)) = WPFDHM\(^{(k)}\)(\(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k\)). When \( x = k \), we can prove it in a similar way. \( \square \)

**Property 15.** *(Boundedness).* Let \( \tilde{a}_i = (\mu_{i_j}, v_{i_j}), \tilde{\alpha}^+ = (\mu_{\maxi}, v_{\maxi}) \) be a set of PFNs, and \( \tilde{\alpha}^- = (\mu_{\minj}, v_{\minj}) \), and weight vector meets \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \) then
\[
\tilde{\alpha}^- \leq \text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \leq \tilde{\alpha}^+ \quad (55)
\]

**Proof.** Based on Properties 13 and 14, we have
\[
\text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \geq \text{WPFDHM}^{(k)}_{\omega}(\tilde{\alpha}^-, \tilde{\alpha}^-, \ldots, \tilde{\alpha}^-) = \tilde{\alpha}^-,
\]
\[
\text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \leq \text{WPFDHM}^{(k)}_{\omega}(\tilde{\alpha}^+, \tilde{\alpha}^+, \ldots, \tilde{\alpha}^+) = \tilde{\alpha}^+.
\]

Then we have \( \tilde{\alpha}^- \leq \text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \leq \tilde{\alpha}^+ \). \( \square \)

**Property 16.** *(Commutativity).* Let \( \tilde{a}_i = (\mu_{i_j}, v_{i_j}), \tilde{\alpha}_i = (\mu_{\theta_i}, v_{\theta_i}) \) be two sets of PFNs. Suppose \( (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) \) is any permutation of \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k)\), and weight vector meets \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \), the \( \tilde{\alpha} \) and \( \tilde{\alpha} \) are equal, then we have
\[
\text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \text{WPFDHM}^{(k)}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) \quad (56)
\]

**Proof.** Because \((\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k)\) is any permutation of \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k)\), then
\[
\left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} = \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \quad (1 \leq x < k)
\]
\[
\left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} = \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \sum_{j=1}^{k} \omega_{ij} \right) \left( \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}} \quad (x = k)
\]

Thus \( \text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \text{WPFDHM}^{(k)}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) \). Next, we will discuss some particular cases of WPFDHM operator for different value \( x \).

**Case 1:** When \( x = 1 \), the WPFDHM will reduce to the following form:
\[
\text{WPFDHM}^{(k)}_{\omega}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \prod_{1 \leq i \leq k} \left( \mu_{i_j}^{1 - \omega_{ij}} \right) \right)^{\frac{1}{\omega_i}} \left( \prod_{1 \leq i < j \leq k} \left( 1 - \frac{\tilde{\alpha}^2}{j_i v_j} \right)^{1 - \omega_{ij}} \right)^{\frac{1}{1 - \omega_i}}
\]
Case 2: When \( x = k \), the proposed WPFDHM operator will reduce to the following form:

\[
\text{WPFDHM}_{\omega}^{(k)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) = \left( \sum_{i=1}^{k} \omega_i \right)^{-1} \left( \prod_{i=1}^{k} \left( 1 - (\mu_i)^2 \right)^{1-\omega_i} \right) \]

Example 6. Let \( \tilde{a}_1 = (0.8, 0.2), \tilde{a}_2 = (0.6, 0.3), \tilde{a}_3 = (0.5, 0.2), \tilde{a}_3 = (0.5, 0.4) \) be four PFNs. The weighting vector of attributes be \( \omega = \{0.3, 0.2, 0.4, 0.1\} \). Then we use the proposed WPFDHM operator to aggregate four PHNs. (suppose \( x = 2 \))

\[
\tilde{a} = (\mu, \nu) = \text{WPFDHM}_{\omega}^{(2)}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (0.6334, 0.2636).
\]

4. A MAGDM Approach Based on the Proposed PFHM Operator

In this part, we apply the WPFFH operator to process the MAGDM problem with the information expressed by PFNs. Let \( X = \{x_1, x_2, \ldots, x_m\} \) be a set of alternatives, and \( C = \{c_1, c_2, \ldots, c_n\} \) be a collection attributes, the weighting vector of attributes be \( \omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \), meet \( \omega_i \in [0,1], j = 1,2, \ldots, n, \sum_{i=1}^{n} \omega_i = 1 \). There are experts \( Y = \{y_1, y_2, \ldots, y_z\} \) who are invited to give the evaluation information, and their weighting vector is \( w = \{w_1, w_2, \ldots, w_z\} \) with \( w_b \in [0,1], (b = 1, 2, \ldots, z), \sum_{b=1}^{z} w_b = 1 \). The expert \( y_b \) evaluates each attribute \( c_j \) of each alternative \( x_i \) by the form of PFN \( \tilde{a}_{ij} = (\mu_{ij}, \nu_{ij}) \) \((i = 1, 2, \ldots, m, j = 1, 2, \ldots, n) \), and then the decision matrix \( \bar{A}_b = (\tilde{a}_{ij})_{m \times n} \) \((b = 1, 2, \ldots, z) \) is constructed. The ultimate goal is to give a ranking of all alternatives.

Then, we will give the steps for solving this problem.

Step 1: Based on the WPFFH operator, calculate the collective evaluation value of each attribute for each alternative by \( \bar{d}_{ij} = \text{WPFFH}_{\omega} \left( \tilde{a}_{ij}, \tilde{a}_{ij}, \ldots, \tilde{a}_{ij} \right) \)
Then the decision matrix

Then the decision matrix $\tilde{a}_{ij} = \text{WPFHM}_{\omega}(\tilde{a}_{11}, \tilde{a}_{22}, \ldots, \tilde{a}_{mn})$

Step 3: According to Definition 3, calculate the $S(\tilde{a})$ and $H(\tilde{a})$.

Step 4: Sort all alternatives \{x_1, x_2, \ldots, x_m\} and choose the best one.

5. An Illustrate Example

In this section, we will give an example to explain the proposed method. A company wants to select a supplier and now there are four suppliers as candidates $A_i = (A_1, A_2, A_3, A_4)$. We evaluate each supplier from four aspects $G_i = (G_1, G_2, G_3, G_4)$, which are “production cost”, “production quality”, “supplier’s service performance”, “risk factor”. The weight vector of attributes is $\omega = (0.1, 0.3, 0.4, 0.2)^T$. There are four experts, and the weight vector of the experts is $w = (0.2, 0.4, 0.1, 0.3)^T$. Then the decision matrix $\tilde{R}_b = (\tilde{a}_{ij})_{4 \times 4}$ ($b = 1, 2, 3, 4$) are shown in Tables 1–4, and our goal is to rank four suppliers and select the best one.

Table 1. Decision matrix $\tilde{R}_1$.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.50, 0.30)</td>
<td>(0.40, 0.20)</td>
<td>(0.50, 0.40)</td>
<td>(0.60, 0.50)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.70, 0.20)</td>
<td>(0.60, 0.30)</td>
<td>(0.50, 0.40)</td>
<td>(0.40, 0.30)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.80, 0.10)</td>
<td>(0.60, 0.20)</td>
<td>(0.70, 0.20)</td>
<td>(0.90, 0.10)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.60, 0.50)</td>
<td>(0.30, 0.40)</td>
<td>(0.50, 0.80)</td>
<td>(0.40, 0.70)</td>
</tr>
</tbody>
</table>

Table 2. Decision matrix $\tilde{R}_2$.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.60, 0.50)</td>
<td>(0.70, 0.60)</td>
<td>(0.50, 0.40)</td>
<td>(0.40, 0.20)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.80, 0.10)</td>
<td>(0.40, 0.30)</td>
<td>(0.60, 0.10)</td>
<td>(0.80, 0.30)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.70, 0.40)</td>
<td>(0.80, 0.20)</td>
<td>(0.70, 0.60)</td>
<td>(0.60, 0.20)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.40, 0.60)</td>
<td>(0.50, 0.40)</td>
<td>(0.50, 0.80)</td>
<td>(0.30, 0.60)</td>
</tr>
</tbody>
</table>

Table 3. Decision matrix $\tilde{R}_3$.

<table>
<thead>
<tr>
<th></th>
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<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.40, 0.30)</td>
<td>(0.40, 0.30)</td>
<td>(0.50, 0.70)</td>
<td>(0.40, 0.20)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.50, 0.30)</td>
<td>(0.80, 0.60)</td>
<td>(0.50, 0.70)</td>
<td>(0.60, 0.50)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.80, 0.20)</td>
<td>(0.40, 0.70)</td>
<td>(0.90, 0.10)</td>
<td>(0.60, 0.30)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.50, 0.60)</td>
<td>(0.60, 0.40)</td>
<td>(0.40, 0.60)</td>
<td>(0.50, 0.60)</td>
</tr>
</tbody>
</table>

Table 4. Decision matrix $\tilde{R}_4$.

<table>
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<tr>
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<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.70, 0.60)</td>
<td>(0.30, 0.40)</td>
<td>(0.40, 0.20)</td>
<td>(0.50, 0.30)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.50, 0.80)</td>
<td>(0.70, 0.60)</td>
<td>(0.80, 0.30)</td>
<td>(0.70, 0.20)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.70, 0.50)</td>
<td>(0.60, 0.30)</td>
<td>(0.90, 0.10)</td>
<td>(0.80, 0.30)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.50, 0.70)</td>
<td>(0.30, 0.50)</td>
<td>(0.40, 0.40)</td>
<td>(0.50, 0.40)</td>
</tr>
</tbody>
</table>

5.1. Decision-Making Processes

Step 1: Since the four attributes are of the same type, we don’t need to normalize the matrix $\tilde{R}_1 \sim \tilde{R}_4$.

Step 2: Use WPFHM operator to aggregate four decision matrix $\tilde{R}_b = (\tilde{a}_{ij})_{m \times n}$ into a collective matrix $\tilde{R} = (\tilde{a}_{ij})_{m \times n}$ which is listed in Table 5 (suppose $x = 2$).
Table 5. The collective decision matrix $\bar{R}$.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.3857, 0.1549)</td>
<td>(0.4084, 0.2074)</td>
<td>(0.3622, 0.1334)</td>
<td>(0.3695, 0.1728)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.4071, 0.1010)</td>
<td>(0.4373, 0.0533)</td>
<td>(0.4257, 0.3014)</td>
<td>(0.4497, 0.2756)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.4673, 0.0270)</td>
<td>(0.4557, 0.1155)</td>
<td>(0.4483, 0.1293)</td>
<td>(0.4683, 0.1242)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.3657, 0.3949)</td>
<td>(0.3432, 0.3752)</td>
<td>(0.3893, 0.3329)</td>
<td>(0.3551, 0.2813)</td>
</tr>
</tbody>
</table>

Use WPFDHM operator to aggregate four decision matrixes $\bar{R}_b = (\tilde{a}_{ij}^b)_{m \times n}$ into a collective matrix $\bar{R} = (\tilde{a}_{ij})_{m \times n}$ which is shown in Table 6 (suppose $x = 2$).

Table 6. The collective decision matrix $\bar{R}$.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>$G_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.2751, 0.4911)</td>
<td>(0.3346, 0.6162)</td>
<td>(0.2014, 0.5589)</td>
<td>(0.2609, 0.5144)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.3357, 0.4469)</td>
<td>(0.4803, 0.2891)</td>
<td>(0.3954, 0.7189)</td>
<td>(0.4951, 0.6645)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.6078, 0.2394)</td>
<td>(0.5182, 0.5385)</td>
<td>(0.5584, 0.4363)</td>
<td>(0.6114, 0.3937)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.2323, 0.7933)</td>
<td>(0.1602, 0.8023)</td>
<td>(0.2737, 0.7366)</td>
<td>(0.2046, 0.6824)</td>
</tr>
</tbody>
</table>

Step 3: Use the WPFDHM (WPFDHM) operator to fuse all the attribute values $\tilde{a}_{ij}, \tilde{a}_{ij}(j = 1, 2, 3, 4)$ and get the comprehensive evaluation value (suppose $x = 2$).

$$\tilde{a}_1 = (0.3015, 0.0138), \tilde{a}_2 = (0.3021, 0.0370), \tilde{a}_3 = (0.3083, 0.0443), \tilde{a}_4 = (0.3003, 0.0868).$$

$$\tilde{a}_1 = (0.0849, 0.7369), \tilde{a}_2 = (0.2034, 0.7190), \tilde{a}_3 = (0.5482, 0.5772), \tilde{a}_4 = (0.0573, 0.9129).$$

Step 4: Obtain the score values.

$$S(\tilde{a}_1) = 0.5382, S(\tilde{a}_2) = 0.5454, S(\tilde{a}_3) = 0.5508, S(\tilde{a}_4) = 0.5272.$$  
$$S(\tilde{a}_{12}) = 0.2321, S(\tilde{a}_{12}) = 0.2622, S(\tilde{a}_{12}) = 0.3940, S(\tilde{a}_{12}) = 0.0849.$$  

Step 5: Rank all alternatives $\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$, then the best choice is $\tilde{a}_3$.

Considering the different parameter values in the WPFDHM operator may have an impact on the ordering results, so we calculate the scores produced from the different $x$ and the results are shown in Table 7.

Table 7. Score and ranking of the alternatives with different parameter values $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Score of $S(\tilde{a}_i)$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 1$</td>
<td>$S(\tilde{a}_1) = 0.5693, S(\tilde{a}_2) = 0.5454, S(\tilde{a}_3) = 0.5473.$</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>$S(\tilde{a}_1) = 0.5382, S(\tilde{a}_2) = 0.5454, S(\tilde{a}_3) = 0.5508, S(\tilde{a}_4) = 0.5272.$</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>$x = 3$</td>
<td>$S(\tilde{a}_1) = 0.6529, S(\tilde{a}_2) = 0.8097, S(\tilde{a}_3) = 0.9126, S(\tilde{a}_4) = 0.6028.$</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>$x = 4$</td>
<td>$S(\tilde{a}_1) = 0.5251, S(\tilde{a}_2) = 0.5794, S(\tilde{a}_3) = 0.7034, S(\tilde{a}_4) = 0.4030.$</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
</tbody>
</table>

Considering the different parameter values in the WPFDHM operator may have an impact on the ordering results, so we calculate the scores produced from the different $x$ and the results are shown in Table 8.
Table 8. Score and order of the alternatives with different parameter values $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Score of $S(\tilde{a}_i)$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$S(\tilde{a}_1) = 0.5251$, $S(\tilde{a}_2) = 0.5794$, $S(\tilde{a}_3) = 0.7034$, $S(\tilde{a}_4) = 0.4030$.</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>2</td>
<td>$S(\tilde{a}_1) = 0.2321$, $S(\tilde{a}_2) = 0.2622$, $S(\tilde{a}_3) = 0.3940$, $S(\tilde{a}_4) = 0.0849$.</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>3</td>
<td>$S(\tilde{a}_1) = 0.0583$, $S(\tilde{a}_2) = 0.0861$, $S(\tilde{a}_3) = 0.1941$, $S(\tilde{a}_4) = 0.0018$.</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>4</td>
<td>$S(\tilde{a}_1) = 0.8042$, $S(\tilde{a}_2) = 0.9026$, $S(\tilde{a}_3) = 0.9639$, $S(\tilde{a}_4) = 0.6632$.</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
</tbody>
</table>

From Tables 7 and 8, we can get following conclusions.

When $x = 1$, the sorting of alternatives is $\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$, and the best choice is $\tilde{a}_3$.

When $x = 2, 3, 4$, the sorting of alternatives is $\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$, and the best choice is $\tilde{a}_3$.

5.2. Comparative Analysis

Then, we compare our proposed method with PFWA operator and PFWG operator [60] and the comparative results are listed in Table 9.

Table 9. Ordering of the green suppliers.

<table>
<thead>
<tr>
<th></th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFWA</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
<tr>
<td>PFWG</td>
<td>$\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_1 \succ \tilde{a}_4$</td>
</tr>
</tbody>
</table>

From Table 9, we can get the same ranking results. However, the PFWA operator and PFWG operator fail to consider the relationship between arguments. Our proposed WPFHM and WPFDHM operators capture the relationship among arguments being aggregated.

6. Conclusions

For this paper, we investigate the MADM problems with PFNs. Then, we utilize the HM operator, DHM operator, WHM operator and WDHM operator to develop the PFHM operator, WPFFHM operator, PFDHM operator and WPFDHM operator. The prominent properties of these operators are analyzed. Then, we develop some methods to solve the MADM problems with PFNs. Finally, a practical example for supplier selection is given. In our subsequent works, the extension and application of these operators of PFNs needs to be investigated in other MADM [61,62], risk analysis and uncertain contexts [63,64].

Author Contributions: Z.L., G.W. and M.L. conceived and worked together to achieve this work, G.W. compiled the computing program by Matlab and analyzed the data, Z.L. and G.W. wrote the paper. Finally, all the authors have read and approved the final manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

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