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Fibonacci and Lucas Numbers of the Form $2^a + 3^b + 5^c + 7^d$

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Abstract: In this paper, we find all Fibonacci and Lucas numbers written in the form $2^a + 3^b + 5^c + 7^d$, in non-negative integers a, b, c, d, with $0 \leq \max\{a, b, c\} \leq d$.

Keywords: Fibonacci; Lucas; linear form in logarithms; continued fraction; reduction method

MSC: 11B39; 11J86; 11D61

1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence which is a second-order linear recursive sequence given by $F_{n+2} = F_{n+1} + F_n$, its initial values are $F_0 = 0$ and $F_1 = 1$, and its companion Lucas sequence $\{L_n\}_{n\geq 0}$ follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$. Fibonacci and Lucas numbers are very famous because they have amazing features (consult [1–3]). The problem of looking for a specific form of second-order recursive sequence has a very rich history. Bugeaud, Mignotte and Siksek [4] showed that $0, 1, 8, 144$ and $1, 4$ are the only Fibonacci and Lucas numbers, respectively, of the form y^t with $t > 1$ (perfect power). Other related papers searched for Fibonacci numbers of forms such as $px^2 + 1, px^3 + 1$ [5], $k^2 + k + 2$ [6], $p^a \pm p^b + 1$ [7]. In 1989, Luo [8] solved Vern Hoggatt's conjecture and proved that the only triangle numbers in the Fibonacci sequence $\{F_n\}$ are $1, 3, 21, 55$. In 1991, Luo [9] found all triangular numbers in the Lucas sequence $\{L_n\}$. In [10], Eric F. Bravo and Jhon J. Bravo found all positive integer solutions of the Diophantine equation $F_n + F_m + F_l = 2^a$ in non-negative integers n, m, l, and a with $n \geq m \geq l$. In [11], Normenyo, Luca and Togbé determined all base-10 repdigits that are expressible as sums of four Fibonacci or Lucas numbers. In [12], Marques and Togbé searched for Fibonacci numbers of the form $2^a + 3^b + 5^c$ which are sum of three perfect powers of some prescribed distinct bases.

In this paper, we are interested in Fibonacci numbers and Lucas numbers which are sum of four perfect powers of several prescribed distinct bases. The number of perfect powers involved in the Diophantine equation solved by the literature [12] is one less than the perfect powers involved in the equation solved by us and the amount of computation in the literature [12] is relatively small. More precisely, our results are the following.

Theorem 1. *The solutions of the Diophantine equation*

$$F_n = 2^a + 3^b + 5^c + 7^d \tag{1}$$

in non-negative integers n, a, b, c, d *with* $0 \leq \max\{a, b, c\} \leq d$ *are* $(n, a, b, c, d) \in \{(7, 1, 1, 0, 1), (10, 1, 1, 0, 2), (10, 2, 0, 0, 2), (14, 1, 3, 1, 3), (14, 3, 0, 2, 3)\}$.

Theorem 2. *The solutions of the Diophantine equation*

$$L_n = 2^a + 3^b + 5^c + 7^d \tag{2}$$

in non-negative integers n, a, b, c, d *with* $0 \leq \max\{a, b, c\} \leq d$ *are* $(n, a, b, c, d) \in \{(3, 0, 0, 0, 0), (5, 1, 0, 0, 1), (9, 0, 0, 2, 2)\}$.

2. Preliminaries

Before proceeding further, we recall some facts and tools which will be used later.

First, we recall the Binet's formulae for Fibonacci and Lucas sequences:

$$F_n = \frac{\gamma^n - \mu^n}{\gamma - \mu}$$

and

$$L_n = \gamma^n + \mu^n,$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$ are the roots of $F_n' s$ characteristic polynomial $x^2 - x - 1 = 0$. For all positive integers n, the inequalities

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1}, \quad \gamma^{n-1} \leq L_n \leq 2\gamma^n \tag{3}$$

hold.

In order to prove our theorem, one tool used is a Baker type lower bound for a linear form in logarithms of algebraic numbers, and such a bound was given by the following result of Matveev (see [13]).

Lemma 1. *Let* $\gamma_1, \gamma_2, \cdots, \gamma_t$ *be real algebraic numbers and let* $b_1, \cdots, b_t$ *be non-zero rational integers. Let* D *be the degree of the number field* $\mathbb{Q}(\gamma_1, \gamma_2, \cdots, \gamma_t)$ *over* $\mathbb{Q}$ *and let* A_j *be a real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\}$$

for $j = 1, \cdots, t$. *Assume that*

$$B \geq \max\{|b_1|, \cdots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

As usual, in the above statement, the logarithmic height of an $s-$degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}(\log|a| + \sum_{j=1}^{s} \log\max\{1, |\gamma^{(j)}|\},$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $\gamma^{(j)}, 1 \leq j \leq s$ are the conjugates of γ (over $\mathbb{Q}$).

After finding an upper bound on n which is in general too large, the next step is to reduce it. For that, we need a variant of the famous Baker–Davenport lemma which was developed by Dujella and Pethő [14]. For a real number x, we use $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 2. *(see [10]) Let* M *be a positive integer, let* $\frac{p}{q}$ *be a convergent of the continued fraction of the irrational number* α *such that* $q > 6M$, *and let* A, B, τ *be some real numbers with* $A > 0$ *and* $B > 1$.

Let $\epsilon := \|\tau q\| - M\|\alpha q\|$, *where* $\| \cdot \|$ *denotes the distance from the nearest integer. If* $\epsilon > 0$, *then no solution to the inequality*

$$0 < |u\alpha - v + \tau| < AB^{-\omega}$$

exists in positive integers u, v, *and* ω *with* $u \leq M$ *and* $w \geq \frac{\log(Aq/\epsilon)}{\log B}$.

Next, we are ready to handle the proofs of our results.

3. Proof of Theorem 1

3.1. Bounding n

By combining the Binet formula together with (1), we get

$$\frac{\gamma^n}{\sqrt{5}} - 7^d = 2^a + 3^b + 5^c + \frac{\mu^n}{\sqrt{5}} > 0, \tag{4}$$

because $|\mu| < 1$ while $2^a \geq 1$. Thus,

$$\frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1 = \frac{2^a}{7^d} + \frac{3^b}{7^d} + \frac{5^c}{7^d} + \frac{\mu^n}{7^d\sqrt{5}} > 0 \tag{5}$$

yields

$$\left| \frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1 \right| < \frac{4}{7^{0.1d}}. \tag{6}$$

From the first inequality of (3), we obtain the estimate $\gamma^{n-2} < 4 \times 7^d$ and $7^d < \gamma^{n-1}$, which implies that $0.24n - 1.9 < d < 0.25(n-1)$; also, this yields $d < n$.

We are in a situation where we can apply Matveev's result Lemma 1 to the left side of (6). The left expression of (6) is nonzero, since, if this expression is zero, it means that $\gamma^{2n} = 7^{2d} \times 5 \in \mathbb{Z}$, so $\gamma^{2n} \in \mathbb{Z}$ for some positive integer n, which is false. We take $t := 3, \gamma_1 := \gamma, \gamma_2 := 7, \gamma_3 := \sqrt{5}$ and $b_1 := n, b_2 := -d, b_3 := -1$. Then we have $D = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. Note that $h(\gamma_1) = \frac{1}{2}\log\gamma, h(\gamma_2) = \log 7$ and $h(\gamma_3) = \log\sqrt{5}$. Thus, we can take $A_1 := 0.5, A_2 := 3.9$ and $A_3 := 1.7$. Note that $\max\{|b_1|, |b_2|, |b_3|\} = \max\{n, d, 1\} = n$. We are in position to apply Matveev's result Lemma 1. This lemma together with a straightforward calculation gives

$$\left| \frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1 \right| > \exp(-C(1 + \log n)), \tag{7}$$

where $C = 3.22 \times 10^{12}$. Thus, from (6), (7) and $d > 0.24n - 1.9$, taking logarithms in the inequalities (6), (7) and comparing the resulting inequalities, we get

$$0.046n - 1.8 < 3.22 \times 10^{12} \times (1 + \log n),$$

giving $n < 2.56 \times 10^{15}$. We summarize the conclusions of this section as follows.

Lemma 3. *If* (n, a, b, c, d) *is a solution in positive integers to Equation* (1) *with* $0 \leq \max\{a, b, c\} \leq d$, *then*

$$d < n < 2.56 \times 10^{15}.$$

3.2. Reducing the Bound on n

We use Lemma 2 several times to reduce the bound for n. We return to (6). Put

$$\Lambda_F := n\log\gamma - d\log 7 - \log\sqrt{5}.$$

Then (5), (6) implies that

$$0 < \Lambda_F < e^{\Lambda_F} - 1 < \frac{4}{7^{0.1d}}. \tag{8}$$

Dividing across by log7, we get

$$0 < |n\frac{\log\gamma}{\log 7} - d - \frac{\log\sqrt{5}}{\log 7}| < \frac{2.1}{7^{0.1d}}. \tag{9}$$

We are now ready to apply Lemma 2 with the obvious parameters,

$$\alpha := \frac{\log\gamma}{\log 7}, v := d, \tau := -\frac{\sqrt{5}}{\log 7}, A := 2.1, B := 1.2.$$

It is easy to see that α is irrational. In fact, we assume that $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ and $\gcd(p,q) = 1$. Then $\gamma^q = 7^p$, hence $\overline{\gamma}^q = 7^p$, where $\overline{\gamma}$ is the conjugate of γ. Thus, we can get $\gamma^q\overline{\gamma}^q = 7^{2p}$; hence, $(-1)^q = 7^{2p}$ which is an absurdity. We can take $M := 2.56 \times 10^{15}$. Let $\frac{p_k}{q_k}$ be the kth convergent of the continued fraction of α. By applying Lemma 2 and performing the calculations with $q_{39} > 6M$ and $\epsilon = \|\tau q_{39}\| - M\|\alpha q_{39}\| = 0.42904\cdots$, we get that if (n, a, b, c, d) is a solution in positive integers of Equation (1), then $d < 225$, which implies that

$$n < \frac{226.9}{0.24} = 945.417 < 946.$$

Then we can take $M := 946$. By applying Lemma 2 again and performing the calculations with $q_8 > 6M$ and $\epsilon = \|\tau q_8\| - M\|\alpha q_8\| = 0.07417\cdots$, we get that if (n, a, b, c, d) is a solution in positive integers of Equation (1), then $d < 73$, which implies that

$$n < 313.$$

Finally, we apply a program written in Mathematica to determine the solutions to (1) in the range $0 \le \max\{a, b, c\} \le d < 73$ and $n < 313$. Quickly, the program returns the following solutions: $(n, a, b, c, d) \in \{(7,1,1,0,1), (10,1,1,0,2), (10,2,0,0,2), (14,1,3,1,3), (14,3,0,2,3)\}$. This proof has been completed.

4. Proof of Theorem 2

4.1. Bounding n

By combining Binet formula together with (2), we get

$$\gamma^n - 7^d = 2^a + 3^b + 5^c - \mu^n > 0, \tag{10}$$

because $|\mu| < 1$ while $2^a \ge 1$. Thus,

$$\gamma^n 7^{-d} - 1 = \frac{2^a}{7^d} + \frac{3^b}{7^d} + \frac{5^c}{7^d} - \mu^n 7^{-d} > 0 \tag{11}$$

yields

$$|\gamma^n 7^{-d} - 1| < \frac{4}{7^{0.1d}}. \tag{12}$$

From the second inequality of (3) and (2) , we obtain the estimate $\gamma^{n-1} < 4 \times 7^d$ and $7^d < 2 \times \gamma^n$, which implies that $4.04d - 1.45 < n < 4.05d + 3.89$; also, this yields $d \le n$.

We are also in a situation where we can apply Matveev's result Lemma 1 to the left side of (12). The left expression of (12) is nonzero, since, if this expression is zero, it means that $\gamma^n = 7^d \in \mathbb{Z}$, so $\gamma^n \in \mathbb{Z}$ for some positive integer n, which is false. We take $t := 2$, $\gamma_1 := \gamma, \gamma_2 := 7$ and $b_1 := n, b_2 := -d$.

Then we have $D = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. Note that $h(\gamma_1) = \frac{1}{2}\log\gamma, h(\gamma_2) = \log 7$. Thus, we can take $A_1 := 0.5, A_2 := 3.9$. Note that $B = \max\{|b_1|, |b_2|\} = \max\{n, d\} = n$. We are in position to apply Matveev's result Lemma 1. This lemma together with a straightforward calculation gives

$$| \gamma^n 7^{-d} - 1 |> \exp(-C(1 + \log n)), \tag{13}$$

where $C = 1.02 \times 10^{10}$. Thus, from (12), (13) and $d > \frac{n-3.89}{4.05}$, taking logarithms in the inequalities (12), (13) and comparing the resulting inequalities, we get

$$0.1(n-1)\log\gamma - 1.1 \times \log 4 < C \times (1 + \log n),$$

giving $n < 6.47 \times 10^{12}$. The conclusions of this section are as follows.

Lemma 4. *If (n, a, b, c, d) is a solution in positive integers to Equation (2) with $0 \leq \max\{a, b, c\} \leq d$, then*

$$d \leq n < 6.47 \times 10^{12}.$$

4.2. Reducing the Bound on n

We use the extremality property of continued fraction to reduce the bound for n. We return to (12) and put

$$\Lambda_L := n\log\gamma - d\log 7.$$

Then (11), (12) implies that

$$0 < \Lambda_L < e^{\Lambda_L} - 1 < \frac{4}{7^{0.1d}}. \tag{14}$$

Dividing by log7, we get

$$0 < n\frac{\log\gamma}{\log 7} - d < \frac{2.1}{1.2^d}. \tag{15}$$

Let $[a_0, a_1, a_2, a_3, a_4, \cdots,] = [0, 4, 22, 1, 5, 1, 1, 17, \cdots]$ be the continued fraction of $\frac{\log\gamma}{\log 7}$, and let $\frac{p_k}{q_k}$ be its kth convergent. Recall that $n < 6.47 \times 10^{12}$ by Lemma 4. A quick inspection using Mathematica reveals that $q_{19} < 1.662 \times 10^{12} < q_{20}$. Furthermore, $a_M := \max\{a_i : i = 0, 1, \cdots, 27\} = a_{14} = 35$. So, in accordance with the extremality property of continued fraction, we obtain that

$$|n\frac{\log\gamma}{\log 7} - d| > \frac{1}{(a_M + 2)n} = \frac{1}{37n}. \tag{16}$$

By comparing estimates (15) and (16), we get right away that

$$\frac{1}{37n} < \frac{2.1}{1.2^d}.$$

This leads to

$$d < \frac{\log(2.1 \times 37n)}{\log 1.2} < 186,$$

which implies that

$$n < 757.$$

This can lead to

$$d < \frac{\log(2.1 \times 37n)}{\log 1.2} < 61,$$

which implies that

$$n < 251.$$

Finally, we use a program written in Mathematica to find the solutions to (2) in the range $0 \leq \max\{a, b, c\} \leq d < 61$ and $n < 251$. Quickly, the program returns the following solutions: $(n, a, b, c, d) \in \{(3, 0, 0, 0, 0), (5, 1, 0, 0, 1), (9, 0, 0, 2, 2)\}$. This completes the proof.

5. Conclusions

In this paper, we find all the solutions of the Diophantine equation (1) by using a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers and the Lemma 2 from Diophantine approximation to reduce the upper bounds on the variables of the equation. For the Diophantine equation (2), we solve the equation by using the lower bound for a nonzero linear form in logarithms of algebraic numbers and the extremality properties of continued fraction to reduce the upper bounds on the variables of the equation.

6. Future Developments

We remark that we can further take advantage of our method to prove that there are only finitely many solutions (and all of them are effectively computable) for the Diophantine equation $F_n = -2^a - 3^b - 5^c + 7^d, L_n = -2^a - 3^b - 5^c + 7^d$ in non-negative integers n, a, b, c, d with $0 \leq \max\{a, b, c\} \leq d$. We leave this as a problem for other researchers.

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