Article

Geometric Properties of Normalized Mittag–Leffler Functions

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Abstract: The aim of this paper is to investigate certain properties such as convexity of order \( \mu \), close-to-convexity of order \( (1 + \mu)/2 \) and starlikeness of normalized Mittag–Leffler function. We use some inequalities to prove our results. We also discuss the close-to-convexity of Mittag–Leffler functions with respect to certain starlike functions. Furthermore, we find the conditions for the above-mentioned function to belong to the Hardy space \( \mathcal{H}^p \). Some of our results improve the results in the literature.

Keywords: analytic functions; Mittag–Leffler functions; starlike functions; convex functions; Hardy space

MSC: 30C45; 33E12.

1. introduction

The one parameter Mittag–Leffler function \( E_\alpha(z) \) defined by

\[
E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}
\]  

was introduced by Mittag–Leffler [1]. This function of complex variable is entire. The series defined by Equation (1) converges in \( \mathbb{C} \) when \( \text{Re}(\alpha) > 0 \). Consider that the function \( E_{\alpha,\kappa}(z) \) which generalizes the function \( E_\alpha(z) \) is defined by

\[
E_{\alpha,\kappa}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \kappa)}, \alpha, \kappa \in \mathbb{C}, z \in \mathbb{C}.
\]

It was introduced by Wiman [2] and was named as Mittag–Leffler type function. The series in Equation (2) converges in \( \mathbb{C} \) when \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\kappa) > 0 \). Furthermore, the functions defined in (1) and (2) are entire of order \( 1/\text{Re}(\alpha) \) and of type 1, for more details, see [3]. The function \( E_{\alpha,\kappa}(z) \) and its analysis with its generalizations is increasingly becoming a rich research area in mathematics and its related fields. A number of researchers studied and analyzed the function given in (2) (see Wiman [2,4,5]). One can find this function in the study of kinetic equation of fractional order, Lévy flights, random walks, super-diffusive transport as well as in investigations of complex systems.
In a similar manner, the advanced studies of these functions reflect and highlight many vital properties of these functions. The function \( E_{\alpha,\kappa} (z) \) generalizes many functions such as

\[
\begin{align*}
E_{1,1}(z) &= e^z, \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \\
E_{2,1}(z) &= \cosh(\sqrt{z}), \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.
\end{align*}
\]

The interested readers are suggested to go through [6–9].

Let \( \mathcal{A} \) be the family of all functions \( g \) having the form

\[
g(z) = z + \sum_{m=2}^{\infty} a_m z^m,
\]

and are analytic in \( D = \{ z : |z| < 1 \} \) and \( S \) denote the family of univalent functions from \( \mathcal{A} \). The families of functions which are convex, starlike and close-to-convex of order \( \mu \), respectively, are defined as:

\[
\begin{align*}
C(\mu) &= \left\{ g : g \in \mathcal{A} \text{ and } \Re \left( 1 + \frac{zg''(z)}{g'(z)} \right) > \mu, \quad z \in D; \quad 0 \leq \mu < 1 \right\}, \\
S^*(\mu) &= \left\{ g : g \in \mathcal{A} \text{ and } \Re \left( \frac{zg'(z)}{g(z)} \right) > \mu, \quad z \in D; \quad 0 \leq \mu < 1 \right\}, \\
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{K}(\mu) &= \left\{ g : g \in \mathcal{A} \text{ and } \Re \left( \frac{g'(z)}{h'(z)} \right) > \mu, \quad z \in D; \quad 0 \leq \mu < 1; \quad h \in C \right\}.
\end{align*}
\]

It is clear that \( C^*(0) = C, S^*(0) = S^* \) and \( \mathcal{K}(0) = \mathcal{K} \). Consider the class \( \mathcal{H} \) of all analytic functions in \( D \) and \( \mu < 1 \), Baricz [10] introduced the classes

\[
\begin{align*}
\mathcal{P}_\eta(\mu) &= \left\{ p : p \in \mathcal{H}, p(0) = 1, \Re \left\{ e^{\eta} \left( p(z) - \mu \right) \right\} > 0, \quad z \in D, \quad \eta \in \mathbb{R} \right\}, \\
\mathcal{R}_\eta(\mu) &= \left\{ f : f \in \mathcal{A} \text{ and } \Re \left\{ e^{\eta} \left( f'(z) - \mu \right) \right\} > 0, \quad z \in D, \quad \eta \in \mathbb{R} \right\}.
\end{align*}
\]

For \( \eta = 0 \), we have the classes of analytic functions \( \mathcal{P}_0(\alpha) \) and \( \mathcal{R}_0(\alpha) \) respectively. Also for \( \eta = 0 \) and \( \alpha = 0 \), we have the classes \( \mathcal{P} \) and \( \mathcal{R} \).

For the functions \( g \in \mathcal{A} \) given by (1) and \( h \in \mathcal{A} \) given by

\[
h(z) = z + \sum_{m=2}^{\infty} b_m z^m,
\]

then the convolution (Hadamard product) of \( g \) and \( h \) is defined as:

\[
(g * h)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \quad z \in D.
\]

It is clear that the function \( E_{\alpha,\kappa} (z) \) is not in class \( \mathcal{A} \). Recently, Bansal and Prajapat [11] considered the normalization of the function \( E_{\alpha,\kappa} (z) \) given as

\[
E_{\alpha,\kappa}(z) = \Gamma(\kappa) z E_{\alpha,\kappa}(z) = z + \sum_{m=1}^{\infty} \frac{\Gamma(k)}{\Gamma(a m + k)} \frac{z^{m+1}}{a m + k}, \quad a, \kappa \in \mathbb{C}, \quad \Re(\alpha) > 0, \kappa \neq 0, -1, \cdots.
\]
In this article, we investigate some geometric properties of function \( E_{a, \kappa}(z) \) with real parameters \( a \) and \( \kappa \).

We need the following results in our investigations.

**Lemma 1** ([12]). If \( g \in A \) and

\[
|zg''(z)| < \frac{1 - \mu}{4}, \quad z \in D; \quad 0 \leq \mu < 1,
\]

then

\[
\text{Re} \{ g'(z) \} > \frac{1 + \mu}{2}, \quad z \in D; \quad 0 \leq \mu < 1.
\]

**Lemma 2** ([13]). Let \( \kappa \in \mathbb{C} \) such that \( \text{Re}(\kappa) > 0, c \in \mathbb{C} \) and \( |c| \leq 1, \ c \neq -1 \). If \( h \in A \) satisfies

\[
|cz|^{2\beta} + \left( 1 - |z|^{2\beta} \frac{zh''(z)}{\beta h'(z)} \right) \leq 1, \quad z \in D,
\]

then

\[
C_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} h'(t) dt \right\}^{1/\beta}, \quad z \in D
\]
is analytic and univalent in \( D \).

**Lemma 3** ([14]). Let \( g(z) = z + a_2z^2 + ... + a_mz^m + ..., \) be analytic in \( D \) and in addition \( 1 \geq 2a_2 \geq ... \geq ma_m \geq ... \geq 0 \) or \( 1 \leq 2a_2 \leq ... \leq ma_m \leq ... \leq 2 \), then \( g(z) \) is in class \( K \) with respect to the function \( z \to -\log (1 - z) \). Also if the function \( g(z) = z + 3a_3 + ... + a_{2m-1}z^{2m-1} + ..., \) which is odd and analytic in \( D \) and satisfies in addition \( 1 \geq 3a_3 \geq ... \geq (2m+1)a_{2m+1} \geq ... \geq 0 \) or \( 1 \leq 3a_3 \leq ... \leq (2m+1)a_{2m+1} \leq ... \leq 2 \), then \( g(z) \in S \) in \( D \).

**Lemma 4** ([15]). If \( g(z) = \sum_{m=1}^{\infty} a_mz^{m-1} \), such that \( a_1 = 1 \) and \( a_m \geq 0, \forall m \geq 2 \), is analytic in \( D \) and if \( \{a_m\}_{m=1}^{\infty} \) is a sequence which is decreasing, i.e., \( a_{m+2} + a_m - 2a_{m+1} \geq 0 \) and \( a_m - a_{m+1} \geq 0, \forall m \geq 1 \), then

\[
\text{Re} \left\{ \sum_{m=1}^{\infty} a_mz^{m-1} \right\} > \frac{1}{2}, \quad \forall z \in D.
\]

**Lemma 5** ([15]). If \( a_m \geq 0, \) \( \{ma_m\} \) and \( \{ma_m - (m+1)a_{m+1}\} \) both are non-increasing, then the function \( g \) defined by (3) is in \( S^* \).

2. Starlikeness, Convexity, Close-to-Convexity

**Theorem 1.** Let \( \alpha \geq \frac{3}{2} \) and \( \kappa \geq \frac{3}{2} \). Then,

\[
\text{Re} \left\{ \frac{E_{a, \kappa}(z)}{z} \right\} > \frac{1}{2}, \quad \forall z \in D.
\]

**Proof.** For the proof of this result, we have to show that

\[
\{a_m\}_{m=1}^{\infty} = \left\{ \frac{\Gamma(\kappa)}{\Gamma(\alpha(m-1) + \kappa)} \right\}_{m=1}^{\infty}
\]
is a decreasing sequence. Consider
\[
\begin{align*}
  a_m - a_{m+1} &= \frac{\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)} - \frac{\Gamma(\kappa)}{\Gamma(a(m)+\kappa)} \\
  &= \Gamma(\kappa) \left\{ \frac{\Gamma(a(m)+\kappa) - \Gamma(a(m-1)+\kappa)}{\Gamma(a(m-1)+\kappa)\Gamma(a(m)+\kappa)} \right\} > 0,
\end{align*}
\]

where \( \forall \, m \geq 1, \, \alpha \geq \frac{3}{2} \) and \( \kappa \geq \frac{3}{2} \). Now, to show that \( \{a_m\}_{m=1}^{\infty} \) is decreasing, we prove that \( a_{m+2} \geq 2a_{m+1} \).

Take
\[
\begin{align*}
  a_m - 2a_{m+1} + a_{m+2} &= \frac{\Gamma(\kappa)}{\Gamma(a(m+1)+\kappa)} + \frac{\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)} - 2\frac{\Gamma(\kappa)}{\Gamma(a(m)+\kappa)} \\
  &= \Gamma(\kappa) \left\{ \frac{\Gamma(a(m)+\kappa)\Gamma(a(m)+\kappa) - 2\Gamma(a(m-1)+\kappa)\Gamma(a(m+1)+\kappa)}{\Gamma(a(m-1)+\kappa)\Gamma(a(m)+\kappa)\Gamma(a(m+1)+\kappa)} \right\} \\
  &= \Gamma(\kappa) \left\{ \frac{\Gamma(a(m)+\kappa)\{\Gamma(a(m)+\kappa) - 2\Gamma(a(m-1)+\kappa)\}}{\Gamma(a(m-1)+\kappa)\Gamma(a(m)+\kappa)\Gamma(a(m+1)+\kappa)} \right\} \\
  &= \Gamma(\kappa) \left\{ \frac{\Gamma(a(m)+\kappa)}{\Gamma(a(m-1)+\kappa)\Gamma(a(m)+\kappa)\Gamma(a(m+1)+\kappa)} \right\}.
\end{align*}
\]

The above expression is non negative \( \forall \, m \geq 1, \, \alpha \geq \frac{3}{2} \) and \( \kappa \geq \frac{3}{2} \), which shows that \( \{a_m\}_{m=1}^{\infty} \) is decreasing and convex sequence. Now, from the Lemma 4, we have
\[
\text{Re} \left( \sum_{m=1}^{\infty} b_m z^{m-1} \right) > \frac{1}{2}, \, z \in \mathcal{D},
\]
which is equivalent to
\[
\text{Re} \left( \frac{E_{\alpha,\kappa}(z)}{z} \right) > \frac{1}{2}, \, z \in \mathcal{D}.
\]

\( \Box \)

**Theorem 2.** Let \( \alpha \geq 2.67 \) and \( \kappa \geq 1 \). Then, \( E_{\alpha,\kappa}(z) \) is starlike in the open unit disc \( \mathcal{D} \).

**Proof.** To show that \( E_{\alpha,\kappa}(z) \) is starlike in \( \mathcal{D} \), we prove that \( \{ma_m\} \) and \( \{ma_m - (m+1)a_{m+1}\} \) both are non-increasing in view of Lemma 5. Since \( a_m \geq 0 \) for the normalized Mittag–Leffler function under the given conditions, consider
\[
\begin{align*}
  ma_m - (m+1)a_{m+1} &= \frac{m\Gamma(\kappa)}{\Gamma(a(m)+\kappa)} - \frac{(m+1)\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)} \\
  &= \Gamma(\kappa) \left\{ \frac{m\Gamma(a(m)+\kappa) - (m+1)\Gamma(a(m-1)+\kappa)}{\Gamma(a(m-1)+\kappa)\Gamma(a(m)+\kappa)} \right\} > 0,
\end{align*}
\]
for \( m \geq 1, \, \alpha \geq 2.67 \) and \( \kappa \geq 1 \). Now,
\[
ma_m - 2(m+1)a_{m+1} + (m+2) = \frac{m\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)} - \frac{2(m+1)\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} + \frac{(m+2)\Gamma(\kappa)}{\Gamma(\alpha(m+1)+\kappa)}
\]

\[
= \Gamma(\kappa) \left\{ \frac{-2(m+1)\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha(m+1)+\kappa) + m\Gamma(\alpha m + \kappa)\Gamma(\alpha(m+1)+\kappa) + (m+2)\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m + \kappa)}{\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m + \kappa)\Gamma(\alpha(m+1)+\kappa)} \right\}
\]

\[
= \Gamma(\kappa) \left[ \frac{\Gamma(\alpha(m+1)+\kappa)\{m\Gamma(\alpha m + \kappa) - 2(m+1)\Gamma(\alpha(m-1)+\kappa)\}}{\Gamma(\alpha(m+1)+\kappa)\Gamma(\alpha m + \kappa)\Gamma(\alpha(m+1)+\kappa)} \right].
\]

The above relation is non-negative \(\forall \ m \geq 1, \alpha \geq 2.67 \) and \(\kappa \geq 1\). Thus, from Lemma 5, \(E_{\alpha,\kappa}(z)\) is starlike in \(D\). \(\square\)

**Theorem 3.** Let \(\kappa \geq 3.323\) and \(\kappa \geq 1\). Then,

\[
\Re \{E'_{\alpha,\kappa}(z)\} > \frac{1}{2}, \quad (z \in D).
\]

**Proof.** Consider

\[
E_{\alpha,\kappa}(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\kappa)z^m}{\Gamma(a(m-1)+\kappa)},
\]

\[
E'_{\alpha,\kappa}(z) = 1 + \sum_{m=2}^{\infty} \frac{m\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)}z^{m-1},
\]

\[
E'_{\alpha,\kappa}(z) = 1 + \sum_{m=2}^{\infty} A_m z^{m-1}.
\]

Here, \(A_m = \frac{m\Gamma(\kappa)}{\Gamma(a(m-1)+\kappa)}\). By taking the same computations as in Theorem 2, we get the proof. \(\square\)

**Theorem 4.** If \(\alpha \geq 1\) and \(\kappa \geq 1\), then \(z \to E_{\alpha,\kappa}(z)\) is in \(K\) with respect to the function \(-\log(1-z)\).

**Proof.** Set

\[
E_{\alpha,\kappa}(z) = z + \sum_{m=2}^{\infty} a_{m-1} z^m,
\]

and we have \(a_{m-1} > 0\) for all \(m \geq 2\) and \(a_1 = \frac{\Gamma(\kappa)}{\Gamma(\alpha+\kappa)} \leq 1\). For the proof of this result, we use Lemma 3. Therefore, we have to show that \(\{ma_{m-1}\}_{m \geq 2}\) is decreasing. Now,

\[
ma_{m-1} - (m+1)a_m = \Gamma(\kappa) \left[ \frac{m}{\Gamma(a(m-1)+\kappa)} - \frac{m+1}{\Gamma(\alpha m + \kappa)} \right],
\]

\[
= \Gamma(\kappa) \left[ \frac{m\Gamma(\alpha m + \kappa) - (m+1)\Gamma(\alpha(m-1)+\kappa)}{\Gamma(a(m-1)+\kappa)\Gamma(\alpha m + \kappa)} \right] > 0.
\]

By restricting parameters, we note that \(ma_{m-1} - (m+1)a_m > 0\) for all \(m \geq 2\). Thus, \(\{ma_{m-1}\}_{m \geq 2}\) is a decreasing sequence—hence the result. \(\square\)

**Theorem 5.** If \(\alpha \geq 1\) and \(\kappa \geq 1\), then \(z \to zE_{\alpha,\kappa}(z^2)\) is in \(K\) respect to the function \(\frac{1}{2} \log \left(\frac{1+z}{1-z}\right)\).
Proof. Set

\[ z E_{a,k} \left( z^2 \right) = z + \sum_{m=2}^{\infty} A_{2m-1} z^{2m-1}. \]

Here, \( A_{2m-1} = a_{m-1} = \frac{\Gamma(e)}{\Gamma(a(m-1) + \kappa)} \) for all \( m \geq 2 \). In addition, it is clear that \( a_1 \leq 1 \). Mainly, we have to show that \( \{ (2m-1) a_{m-1} \}_{m \geq 2} \) is decreasing. Now,

\[
(2m-1) a_{m-1} - (2m+1) a_m = \Gamma(k) \left[ \frac{(2m-1)}{\Gamma(a(m-1) + \kappa)} - \frac{(2m+1)}{\Gamma(a(m) + \kappa)} \right],
\]

\[
= \Gamma(k) \left[ \frac{(2m-1) \Gamma(a m + \kappa) - (2m+1) \Gamma(a(m-1) + \kappa)}{\Gamma(a(m-1) + \kappa) \Gamma(a m + \kappa)} \right] > 0.
\]

By using conditions on parameters, we observe that \( (2m-1) a_{m-1} - (2m+1) a_m > 0 \) for all \( m \geq 2 \). Thus, \( \{ (2m-1) a_{m-1} \}_{m \geq 2} \) is a decreasing sequence. By applying Lemma 3, we have the required result. \( \square \)

Theorem 6. If \( \alpha \geq 1 \) and \( \kappa \geq 3.214319744 \), then \( E_{a,k} (z) \in S^* \) in \( D \).

Proof. Let \( p(z) = \frac{z E'_{a,k} (z)}{E_{a,k} (z)}, \ z \in D \). Then, the function \( p \) is analytic in \( D \) with \( p(0) = 1 \). To prove \( E_{a,k} (z) \) is starlike in \( D \), we just prove that \( \Re p(z) > 0 \) in \( z \in D \). For this, it is enough to show \( |p(z) - 1| < 1 \) for \( z \in D \). By using the inequalities

\[
\frac{\Gamma(k)}{\Gamma(a m + \kappa)} \leq \frac{1}{(\kappa)^m}, \ \alpha \geq 1, \ k \geq 1, \ m \in \mathbb{N},
\]

\[
m(k)_m \leq 2^{m-1} k (k+1)^{m-1}, \ k \geq 1, \ m \in \mathbb{N},
\]

we have

\[
\left| E'_{a,k} (z) - \frac{E_{a,k} (z)}{z} \right| = \left| \sum_{m=1}^{\infty} \frac{\Gamma(k)}{\Gamma(a m + \kappa)} m z^m \right|
\leq \sum_{m=1}^{\infty} \frac{2^{m-1}}{k (k+1)^{m-1}}
\leq \frac{1}{k} \sum_{m=1}^{\infty} \left( \frac{2}{k+1} \right)^{m-1}
= \frac{k+1}{k (k-1)}, \ (k > 1).
\]

Furthermore, using reverse triangle inequality and the inequality \((\kappa)^m \leq (k)^m\), we obtain

\[
\left| E_{a,k} (z) \right| = \left| \sum_{m=1}^{\infty} \frac{\Gamma(k)}{\Gamma(a m + \kappa)} z^m \right|
\geq 1 - \sum_{m=1}^{\infty} \frac{1}{(k+1)_m}
\geq 1 - \frac{1}{k} \sum_{m=1}^{\infty} \left( \frac{1}{k+1} \right)^{m-1}
= \frac{k^2 - k - 1}{k^2} \ (k > 0).
\]
By combining (4) and (5), we get
\[
\left| \frac{zE''_{a,k}(z)}{E_{a,k}(z)} - 1 \right| \leq \frac{\kappa (\kappa + 1)}{(\kappa - 1)(\kappa^2 - \kappa - 1)}. \tag{6}
\]

Therefore, \( E_{a,k}(z) \in S^* \) in \( D \) if \( \frac{(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)} \leq 1 \). In other words, we have to show that \( \kappa^3 - 3\kappa^2 - \kappa + 1 \geq 0 \). The inequality is satisfied for \( \kappa \geq 3.214319744 \). Hence, \( E_{a,k}(z) \) is starlike in \( D \). \( \square \)

**Remark** 1. Recently, Bansal and Prajpat [11] proved that \( E_{a,k}(z) \) is starlike, if \( \alpha \geq 1 \) and \( \kappa \geq (3 + \sqrt{7})/2 \approx 3.56155281 \). The above result improves the result in [11].

**Theorem 7.** If \( \alpha \geq 1 \) and \( \kappa \geq 3.56155281 \), then \( E_{a,k}(z) \in C \) in \( D \).

**Proof.** Let \( p(z) = 1 + \frac{zE''_{a,k}(z)}{E_{a,k}(z)} \), \( z \in D \). Then, \( p(z) \) is analytic in \( D \) with \( p(0) = 1 \). To show that \( E_{a,k}(z) \) is convex in \( D \), it is enough to prove that \( |p(z) - 1| < 1, z \in D \). By using the inequalities
\[
\frac{\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} \leq \frac{1}{(\kappa/\alpha)_m}, \alpha \geq 1, \kappa \geq 1, m \in \mathbb{N},
\]
\[
2m(m + 1)(\kappa)_m \leq 4^{m-1}\kappa(\kappa + 1)_{m-1}, \kappa \geq 1, m \in \mathbb{N},
\]
we have
\[
\left| zE''_{a,k}(z) \right| = \left| \sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} m(m + 1)z^m \right|
\leq \sum_{m=1}^{\infty} 2 \kappa(\kappa + 1)^{m-1}
\leq \frac{2}{\kappa} \sum_{m=1}^{\infty} \left( \frac{4}{\kappa + 1} \right)^{m-1}
= \frac{2(\kappa + 1)}{\kappa(\kappa - 3)}, \quad (\kappa > 3). \tag{7}
\]
Furthermore, using the inequality \( m(\kappa)^m \leq 2^{m-1}(\kappa)^m \), then we have
\[
\left| E'_{a,k}(z) \right| = \left| 1 + \sum_{m=1}^{\infty} (m + 1) \frac{\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} z^m \right|
\geq 1 - \sum_{m=1}^{\infty} \frac{1}{(\kappa/\alpha)_m}
\geq 1 - \frac{2}{\kappa} \sum_{m=1}^{\infty} \left( \frac{2}{\kappa + 1} \right)^{m-1}
= \frac{\kappa^2 - 3\kappa - 2}{\kappa(\kappa - 1)}, \quad (\kappa > 0). \tag{8}
\]
From (7) and (8), we get
\[
\left| \frac{zE''_{a,k}(z)}{E'_{a,k}(z)} \right| \leq \frac{2(\kappa^2 - 1)}{(\kappa - 1)(\kappa^2 - 3\kappa - 2)}. \tag{9}
\]
This implies that $E_{a,\kappa}(z) \in \mathbb{C}$ in $\mathcal{D}$ if $\frac{2(\kappa^2-1)}{(\kappa-1)(\kappa^2-3\kappa-2)} \leq 1$. To prove our result, we have to show that $\kappa^3 - 6\kappa^2 + 7\kappa + 6 \geq 0$. The inequality is satisfied for $\kappa \geq 3.5615528$. Hence, $E_{a,\kappa}(z)$ is convex in $\mathcal{D}$.  

Consider the integral operator $F_\gamma : \mathcal{D} \to \mathbb{C}$, where $\gamma \in \mathbb{C}, \gamma \neq 0$,

$$F_\gamma (z) = \left\{ \gamma \int_0^z t^{\gamma-2} E_{a,\kappa}(t) \, dt \right\}, \ z \in \mathcal{D}. $$

Here, $F_\gamma \in \mathcal{A}$. We prove that $F_\gamma \in \mathcal{S}$ in $\mathcal{D}$.

**Theorem 8.** Let $M \in \mathbb{R}^+$ such that $|E_{a,\kappa}(z)| \leq M$ in $\mathcal{D}$. If

$$|\gamma - 1| + \frac{\kappa (\kappa + 1)}{(\kappa - 1)(\kappa^2 - \kappa - 1)} + \frac{M}{|\gamma|} \leq 1, $$

then $F_\gamma \in \mathcal{S}$ in $\mathcal{D}$.

**Proof.** A calculation gives

$$\frac{zE''_\gamma (z)}{F'_\gamma (z)} = \frac{zE'_\gamma (z)}{E_{a,\kappa}(z)} + \frac{z^{-1}}{\gamma} E_{a,\kappa}(z) + \gamma - 2, \ z \in \mathcal{D}. $$

Since $E_{a,\kappa}(z) \in \mathcal{A}$, then by Schwarz Lemma, triangle inequality and (6), we obtain

$$(1 - \gamma^2) \frac{zE''_\gamma (z)}{F'_\gamma (z)} \leq (1 - \gamma^2) \left[ |\gamma - 1| + \left| \frac{zE'_\gamma (z)}{E_{a,\kappa}(z)} - 1 \right| + \left| \frac{z^{-1}}{\gamma} E_{a,\kappa}(z) \right| \right] $$

$$\leq (1 - \gamma^2) \left[ |\gamma - 1| + \frac{\kappa (\kappa + 1)}{(\kappa - 1)(\kappa^2 - \kappa - 1)} + \frac{M}{|\gamma|} \right]. $$

By using Lemma 2, $F_\gamma \in \mathcal{S}$ in $\mathcal{D}$.  

**Theorem 9.** Let $a \geq 1, \mu \in [0, 1)$ and $z \in \mathcal{D}$.

(i) If $\kappa \geq \frac{11 - 3\mu + \sqrt{\mu^2 - 12\mu + 17}}{2(1 - \mu)}$, then $E_{a,\kappa}(z) \in \mathcal{K}\left(\frac{1 + \mu}{2}\right)$.

(ii) If $\kappa \leq \frac{11 - 3\mu + \sqrt{\mu^2 - 12\mu + 17}}{2(1 - \mu)}$, then $E_{a,\kappa}(z) \in \mathcal{P}(\mu)$.

(iii) If $(1 - \mu) \kappa^3 + (2\mu - 3) \kappa^2 - \kappa + (1 - \mu) > 0$, then $E_{a,\kappa}(z) \in \mathcal{S}(\mu)$.

(iv) If $(1 - \mu) \kappa^3 + (6\mu - 8) \kappa^2 + (7 - 7\mu) \kappa + (8 - 6\mu) > 0$, then $E_{a,\kappa}(z) \in \mathcal{C}(\mu)$.

**Proof.** (i) Using (7) and Lemma 1, we get

$$|zE''_{a,\kappa}(z)| \leq \frac{2(\kappa + 1)}{\kappa(\kappa - 3)} < 1 - \frac{\mu}{4}, $$

where $0 \leq \mu < 1 - \frac{8(\kappa + 1)}{\kappa(\kappa - 3)}$ and $\kappa > \frac{11 - 3\mu + \sqrt{\mu^2 - 12\mu + 17}}{2(1 - \mu)}$. This shows that $E_{a,\kappa}(z) \in \mathcal{K}\left(\frac{1 + \mu}{2}\right)$.

(ii) To prove $\frac{E_{a,\kappa}(z)}{z} \in \mathcal{P}(\mu)$, we have to show that $|g(z) - 1| < 1$, where $g(z) = \frac{E_{a,\kappa}(z)/z}{(1 - \mu)}$. By using triangle inequality with

$$\frac{\Gamma(\kappa)}{\Gamma(\lambda + \kappa)} \leq \frac{1}{(\kappa)_m}, \ m \in \mathbb{N}, $$

$$(\kappa)_m > \kappa(\kappa + a_0)^{-m-1}, \ (\kappa > 0; m \in \mathbb{N}\setminus\{1, 2\}), $$
(see [16]), where
\[ \alpha_0 \approx 1.302775637 \ldots \]
is the largest root of the equation
\[ \alpha^2 + \alpha - 3 = 0, \]
we have
\[
|g(z) - 1| = \left| \frac{1}{(1 - \mu)} \sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} z^m \right|
\leq \frac{1}{(1 - \mu)} \sum_{m=1}^{\infty} \frac{1}{(\kappa)_m}
\leq \frac{1}{(1 - \mu)} \left\{ \frac{1}{\kappa} + \frac{1}{\kappa (\kappa + 1)} + \sum_{m=3}^{\infty} \frac{1}{\kappa (\kappa + \alpha_0)^{m-1}} \right\}
= \frac{1}{(1 - \mu)} \left\{ (\kappa + 2) (\kappa + \alpha_0) (\kappa + \alpha_0 - 1) + (\kappa + 1) \right\}.
\]
This implies that \( \frac{E_{a,x}(z)}{z} \in \mathcal{P}(\mu) \), for \( 0 < \mu < 1 - \frac{(\kappa + 2) (\kappa + \alpha_0) (\kappa + \alpha_0 - 1) + (\kappa + 1)}{\kappa (\kappa + \alpha_0)^{m-1}} \).

(iii) We use the inequality \( \left| \frac{zE_{a,x}'(z)}{E_{a,x}(z)} - 1 \right| < 1 - \mu \) to show the starlikeness of order \( \mu \) for the function \( E_{a,x}(z) \). By using (4) and (5), we have
\[
\left| \frac{zE_{a,x}'(z)}{E_{a,x}(z)} - 1 \right| \leq \frac{\kappa (\kappa + 1)}{(\kappa - 1) (\kappa^2 - \kappa - 1)} < 1 - \mu.
\]
This implies that
\[ \mu < 1 - \frac{\kappa (\kappa + 1)}{(\kappa - 1) (\kappa^2 - \kappa - 1)}. \]
This completes the proof.

(iv) We use the inequality \( \left| \frac{zE_{a,x}''(z)}{E_{a,x}'(z)} - 1 \right| < 1 - \mu \) to show that \( E_{a,x}(z) \in \mathcal{C}(\mu) \). By using (7) and (8), we have
\[
\left| \frac{zE_{a,x}''(z)}{E_{a,x}'(z)} - 1 \right| \leq \frac{2 (\kappa^2 - 1)}{(\kappa^2 - 3\kappa - 2)} < 1 - \mu.
\]
This implies that
\[ \mu < 1 - \frac{2 (\kappa^2 - 1)}{(\kappa^2 - 3\kappa - 2)}. \]

Substituting \( \mu = 0 \) in Theorem 9, we obtained the following results.

**Corollary 1.** Let \( a \geq 1, z \in D \).

(i) If \( \kappa > \frac{11 + \sqrt{17}}{2} \), then \( E_{a,x}(z) \in \mathcal{K} \left( \frac{1}{2} \right) \).

(ii) If \( \frac{(\kappa + 2)(\kappa + \alpha_0)(\kappa + \alpha_0 - 1) + (\kappa + 1)}{\kappa (\kappa + \alpha_0)(\kappa + \alpha_0 - 1)} < 1 \), then \( E_{a,x}(z) \in \mathcal{P} \).

(iii) If \( \kappa^3 - 3\kappa^2 - \kappa + 1 > 0 \), then \( E_{a,x}(z) \in \mathcal{S}^* \).

(iv) If \( \kappa^3 - 8\kappa^2 + 7\kappa + 8 > 0 \), then \( E_{a,x}(z) \in \mathcal{C} \).

**Remark 2.** It is clear that \( E_{a,x}(z) \in \mathcal{K} \left( \frac{1}{2} \right) \) when \( a \geq 1, \kappa > 7.56155 \) and \( E_{a,x}(z) \in \mathcal{C} \) when \( a \geq 1, \kappa > 6.796963 \). It concludes that our results improve the results of ([17], corollary 2.1).
3. Hardy Space of Mittag–Leffler Function

Consider the class \( \mathcal{H} \) of analytic functions in \( D = \{ z : |z| < 1 \} \) and \( \mathcal{H}^\infty \) denote the space bounded functions on \( \mathcal{H} \). Let \( g \in \mathcal{H} \), set

\[
M_q (r, g) = \left\{ \begin{array}{ll}
\left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^q d\theta \right)^{1/q}, & 0 < q < \infty, \\
\max \{ |g(z)| : |z| \leq r \}, & q = \infty.
\end{array} \right.
\]

If \( M_q (r, g) \) is bounded for \( r \in [0, 1) \), then \( g \in \mathcal{H}^q \). It is clear that

\[
\mathcal{H}^\infty \subset \mathcal{H}^p \subset \mathcal{H}^q, \quad 0 < p < q < \infty.
\]

For some details, see [18]. It is also known [18] that, if \( \text{Re} (g'(z)) > 0 \) in \( D \), then

\[
\left\{ \begin{array}{ll}
g' \in \mathcal{H}^p, & p < 1, \\
g \in \mathcal{H}^{p/(1-p)}, & 0 < p < 1.
\end{array} \right.
\]

Hardy spaces of certain special functions are studied in [10,19,20].

Lemma 6 ([21]). \( \mathcal{P}_0 (\mu) \ast \mathcal{P}_0 (\eta) \subset \mathcal{P}_0 (\gamma) \), where \( \gamma = 1 - 2 (1 - \mu) (1 - \eta) \) and \( \mu, \eta < 1 \). The value \( \gamma \) can not be improved.

Lemma 7 ([22]). For \( \mu, \eta < 1 \) and \( \gamma = 1 - 2 (1 - \mu) (1 - \eta) \), we have \( \mathcal{R}_0 (\mu) \ast \mathcal{R}_0 (\eta) \subset \mathcal{R}_0 (\gamma) \), or equivalently \( \mathcal{P}_0 (\mu) \ast \mathcal{P}_0 (\eta) \subset \mathcal{P}_0 (\gamma) \).

Lemma 8 ([23]). If the function \( g \), convex of order \( \mu \), where \( \mu \in [0, 1) \), is not of the form

\[
g (z) = \left\{ \begin{array}{ll}
l + dz (1 - ze^{i\xi})^{2\mu-1}, & \mu \neq 1/2, \\
l + d \log (1 - ze^{i\xi}), & \mu = 1/2,
\end{array} \right.
\]

for \( d, l \in \mathbb{C} \) and \( \xi \in \mathbb{R} \), then the following statements are true:

(i) There exist \( \delta = \delta (g) > 0 \) such that \( g' \in \mathcal{H}^{\delta+1/2(1-\mu)} \).
(ii) If \( \mu \in [0, 1/2) \) then there exists \( \tau = \tau (g) > 0 \) such that \( g \in \mathcal{H}^{\tau+1/(1-2\mu)} \).
(iii) If \( \mu \geq 1/2 \), then \( g \in \mathcal{H}^\infty \).

Theorem 10. Let \( \mu \in [0, 1), (1 - \mu) \kappa^3 + (6\mu - 8) \kappa^2 + (7 - 7\mu) \kappa + (8 - 6\mu) > 0 \).

(i) If \( \mu \in [0, 1/2) \), then \( E_{\alpha, \kappa} (z) \in \mathcal{H}^{1/(1-2\mu)} \).
(ii) If \( \mu \geq 1/2 \), then \( E_{\alpha, \kappa} (z) \in \mathcal{H}^\infty \).

Proof. By using the definition of the hypergeometric function

\[
_{2}F_{1} (a, b, c; z) = \sum_{m=0}^{\infty} \frac{(a)^m (b)^m (z^m)}{(c)^m m!},
\]

we have

\[
1 + \frac{dz}{(1 - ze^{i\xi})^{1-2\mu}} = 1 + dz_{2F_{1}} (1, 1 - 2\alpha, 1; ze^{i\xi}),
\]

\[
= 1 + d \sum_{m=0}^{\infty} \frac{(1 - 2\alpha)^m e^{i\xi^m} z^{m+1}}{m!}.
\]
for $l, d \in \mathbb{C}, \mu \neq 1/2$ and for real $\xi$. On the other hand,

$$l + d \log \left(1 - ze^{\eta}\right) = l - dz_2 f_1 \left(1, 1, 2; ze^{\xi}\right),$$

$$= l - d \sum_{m=0}^{\infty} \frac{1}{m+1} e^{\xi m} z \eta^{m+1}.$$

Therefore, the function $E_{\alpha, \kappa}(z)$ is not of the form of $l + dz \left(1 - ze^{\eta}\right)^{2\mu-1}$ (for $\mu \neq 1/2$) and $l + d \log \left(1 - ze^{\eta}\right)$ (for $\mu = 1/2$). We know that, by part (iv) of Theorem 9, $E_{\alpha, \kappa}(z) \in C(\mu)$. Therefore, by using Lemma 8, we have the required result. \(\Box\)

**Theorem 11.** Let $l \left(\frac{(k+2)(\alpha_{\mu} + a_0 + (k+a_0-1)+\eta)}{k+1}(k+a_0)(k+a_0-1)\right) < 1$ and $f \in D$. Then, convolution $E_{\alpha, \kappa} * f$ is in $\mathcal{H}^\infty \cap \mathcal{R}$.

**Proof.** Let $h(z) = E_{\alpha, \kappa}(z) * g(z)$. Then, $h'(z) = E_{\alpha, \kappa}(z) * g'(z)$. Using the Corollary 1 part ii, we have $E_{\alpha, \kappa}(z) \in \mathcal{P}$. As $g \in \mathcal{R}$, therefore, by using Lemma 6 $h \in \mathcal{R}$. Now, the function $E_{\alpha, \kappa}(z)$ is complete; therefore, $h(z)$ is complete. This implies that $h(z)$ is bounded. Thus, we have the required result. \(\Box\)

**Theorem 12.** Let $\gamma = 1 - 2(1 - \mu)(1 - \eta)$. When $g \in \mathcal{R}(\gamma)$, then $E_{\alpha, \kappa} * g \in \mathcal{R}(\gamma)$.

**Proof.** Let $h(z) = E_{\alpha, \kappa}(z) * g(z)$. Then, it is clear that $h'(z) = E_{\alpha, \kappa}(z) * g'(z)$. Using Theorem 9 part (ii), we have $E_{\alpha, \kappa}(z) \in \mathcal{P}(\mu)$. As $g \in \mathcal{R}$, therefore, by using Lemma 6 and the fact that $g' \in \mathcal{P}(\eta)$, we have $h'(z) \in \mathcal{P}(\gamma)$, where $\gamma = 1 - 2(1 - \mu)(1 - \eta)$. Consequently, $h \in \mathcal{R}(\gamma)$. \(\Box\)

**Corollary 2.** Let $\mu \in [0, 1)$ and $\gamma < 1 - \frac{2(\alpha_{\mu} + a_0 + (k+a_0-1)+\eta)}{k+1}(k+a_0)(k+a_0-1)$. If $g \in \mathcal{R}(\eta)$, then $E_{\alpha, \kappa} * g \in \mathcal{R}(0)$.

**Corollary 3.** Let $\mu \in [0, 1)$ and $\gamma < 1 - \frac{(k+2)(\alpha_{\mu} + a_0 + (k+a_0-1)+\eta)}{k+1}(k+a_0)(k+a_0-1)$. If $g \in \mathcal{R}(1/2)$, then $E_{\alpha, \kappa} * g \in \mathcal{R}(0)$.

**4. Applications**

Now, we present some applications of the above theorems. It is clear that

$$E_{1,2}(z) = e^z - 1, \quad E_{1,3}(z) = \frac{2e^z - z - 1}{z}, \quad E_{1,4}(z) = \frac{6e^z - 3z^2 - 6z - 6}{z^2}.$$

From Theorem 9, we get the following:

**Corollary 4.** (i) If $0 \leq \mu < \mu_0$, where $\mu_0 \approx 0.26759$, then $E_{1,2}(z) \in \mathcal{P}(\mu)$.

(ii) If $0 \leq \mu < \mu_1$, where $\mu_1 \approx 0.55988$, then $E_{1,3}(z) \in \mathcal{P}(\mu)$.

(iii) If $0 \leq \mu < \mu_2$, where $\mu_2 \approx 0.68904$, then $E_{1,4}(z) \in \mathcal{P}(\mu)$.

**Corollary 5.** If $0 \leq \mu < \mu_3$, where $\mu_3 \approx 0.39393$, then $E_{1,4}(z) \in \mathcal{S}^*(\mu)$.

**Corollary 6.** (i) Let $0 \leq \mu < \mu_4$, where $\mu_4 \approx 0.2675930$. If $g \in \mathcal{R}(\eta)$, $\eta = (1 - 2\mu)/(2 - 2\mu)$, then $E_{1,2}(z) * g \in \mathcal{R}(0)$.

(ii) Let $0 \leq \mu < \mu_5$, where $\mu_5 \approx 0.55987780$. If $g \in \mathcal{R}(\eta)$, $\eta = (1 - 2\mu)/(2 - 2\mu)$, then $E_{1,3}(z) * g \in \mathcal{R}(0)$.

(iii) Let $0 \leq \mu < \mu_6$, where $\mu_6 \approx 0.68904320$. If $g \in \mathcal{R}(\eta)$, $\eta = (1 - 2\mu)/(2 - 2\mu)$, then $E_{1,4}(z) * g \in \mathcal{R}(0)$. 
5. Conclusions

In this paper, we have studied certain geometric properties of Mittag-Leffler functions such as starlikeness, convexity and close-to-convexity. We have also found the Hardy spaces of Mittag-Leffler functions. Further, we have improved some results in the literature.

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