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Chen Inequalities for Warped Product Pointwise Bi-Slant Submanifolds of Complex Space Forms and Its Applications

Akram Ali * and Ali H. Alkhaldi

Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia; ahalkhaldi@kku.edu.sa
* Correspondence: akramali133@gmail.com; Tel.: +966-554-146-618

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Abstract: In this paper, by using new-concept pointwise bi-slant immersions, we derive a fundamental inequality theorem for the squared norm of the mean curvature via isometric warped-product pointwise bi-slant immersions into complex space forms, involving the constant holomorphic sectional curvature $c$, the Laplacian of the well-defined warping function, the squared norm of the warping function, and pointwise slant functions. Some applications are also given.

Keywords: mean curvature; warped products; compact Riemannian manifolds; pointwise bi-slant immersions; inequalities

1. Introduction

In the submanifolds theory, creating a relationship between extrinsic and intrinsic invariants is considered to be one of the most basic problems. Most of these relations play a notable role in submanifolds geometry. The role of immersibility and non-immersibility in studying the submanifolds geometry of a Riemannian manifold was affected by the pioneering work of the Nash embedding theorem [1], where every Riemannian manifold realizes an isometric immersion into a Euclidean space of sufficiently high codimension. This becomes a very useful object for the submanifolds theory, and was taken up by several authors (for instance, see [2–15]). Its main purpose was considered to be how Riemannian manifolds could always be treated as Riemannian submanifolds of Euclidean spaces. Inspired by this fact, Nolker [16] classified the isometric immersions of a warped product decomposition of standard spaces. Motivated by these approaches, Chen started one of his programs of research in order to study the impressibility and non-immersibility of Riemannian warped products into Riemannian manifolds, especially in Riemannian space forms (see [11,17–19]). Recently, a lot of solutions have been provided to his problems by many geometers (see [18] and references therein).

The field of study which includes the inequalities for warped products in contact metric manifolds and the Hermitian manifold is gaining importance. In particular, in [17], Chen observed the strong isometrically immersed relationship between the warping function $f$ of a warped product $M_1 \times_f M_2$ and the norm of the mean curvature, which isometrically immersed into a real space form.

Theorem 1. Let $\tilde{M}(c)$ be a $n$-dimensional real space form and let $\varphi : M = M_1 \times_f M_2$ be an isometric immersion of an $n$-dimensional warped product into $\tilde{M}(c)$. Then:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n^2} ||\nabla H||^2 + n_1 c,$$

(1)
where \( n_i = \dim M_i, i = 1, 2, \) and \( \Delta \) is the Laplacian operator of \( M_1 \) and \( H \) is the mean curvature vector of \( M^n \). Moreover, the equality holds in (1) if, and only if, \( \varphi \) is mixed and totally geodesic and \( n_1 H_1 = n_2 H_2 \) such that \( H_1 \) and \( H_2 \) are partially mean curvatures of \( M_1 \) and \( M_2 \), respectively.

In [2,5,20–31], the authors discuss the study of Einstein, contact metrics, and warped product manifolds for the above-mentioned problems. Furthermore, in regard to the collections of such inequalities, we referred to [12] and references therein. The motivation came from the study of Chen and Uddin [32], which proved the non-triviality of warped-product pointwise bi-slant submanifolds of a Kaehler manifold with supporting examples. If the sectional curvature is constant with a Kaehler metric, then it is called complex space forms. In this paper, we consider the warped-product pointwise bi-slant submanifolds which isometrically immerse into a complex space form, where we then obtain a relationship between the squared norm of the mean curvature, constant sectional curvature, the warping function, and pointwise bi-slant functions. We will announce the main result of this paper in the following.

**Theorem 2.** Let \( \bar{M}^{2m}(c) \) be the complex space form and let \( \varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \bar{M}^{2m}(c) \) be an isometric immersion from warped product pointwise bi-slant submanifolds into \( \bar{M}^{2m}(c) \). Then, the following inequality is satisfied:

\[
\Delta(\ln f) \leq \|\nabla \ln f\|^2 + \frac{n^2}{4n^2} \|H\|^2 + \frac{n_1 c}{4} - \frac{3c}{4n^2} \left( n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2 \right),
\]

where \( \theta_1 \) and \( \theta_2 \) are pointwise slant functions along \( M_1 \) and \( M_2 \), respectively. Furthermore, \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operator on \( M_1^{n_1} \), respectively, and \( H \) is the mean curvature vector of \( M^n \). The equality case holds in (2) if and only if \( \varphi \) is a mixed totally geodesic isometric immersion and the following satisfies

\[
\frac{H_1}{H_2} = \frac{n_2}{n_1}
\]

where \( H_1 \) and \( H_2 \) are the mean curvature vectors along \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively.

As an application of Theorem 2 in a compact orientated Riemannian manifold with a free boundary condition, we prove that:

**Theorem 3.** Let \( M^n = M_1^{n_1} \times_f M_2^{n_2} \) be a compact, orientate warped product pointwise bi-slant submanifold in a complex space form \( \bar{M}^{2m}(c) \) such that \( M_1^{n_1} \) is a \( n_1 \)-dimensional and \( M_2^{n_2} \) is a \( n_2 \)-dimensional pointwise slant submanifold \( \bar{M}^{2m}(c) \). Then, \( M^n \) is simply a Riemannian product if, and only if:

\[
\|H\|^2 \geq \frac{c}{n^2} \left( 3n_1 \cos^2 \theta_1 + 3n_2 \cos^2 \theta_2 - n_1 n_2 \right),
\]

where \( H \) is the mean curvature vector of \( M^n \). Moreover, \( \theta_1 \) and \( \theta_2 \) are pointwise slant functions.

By using classifications of pointwise bi-slant submanifolds which were defined in [32], we derived similar inequalities for warped product pointwise pseudo-slant submanifolds [33], warped product pointwise semi-slant submanifolds [34], and CR-warped product submanifolds [17] in a complex space form as well.

**2. Preliminaries and Notations**

An almost complex structure \( J \) and a Riemannian metric \( g \), such that \( f^2 = -I \) and \( g(JX, JY) = g(X, Y) \), for \( X, Y \in \mathfrak{X}(M) \), where \( I \) denotes the identity map and \( \mathfrak{X}(M) \) is the space containing vector fields tangent to \( \bar{M} \), then \( (M, J, g) \) is an almost Hermitian manifold. If the almost complex structure
Weingarten formulas are defined by
\[ \tilde{R}(U, V, Z, W) = \frac{c}{4} \left( g(U, Z)g(V, W) - g(V, Z)g(U, W) + g(U, JZ)g(JV, W) 
\right. \\
\left. - g(V, JZ)g(U, JW) + 2g(U, JV)g(JZ, W) \right), \quad (4) \]
for every \( U, V, Z, W \in \mathfrak{X}(\tilde{M}^{2m}(c)) \). A Riemannian manifold \( \tilde{M}^m \) and its submanifold \( M \), the Gauss and Weingarten formulas are defined by \( \tilde{\nabla}_U V = \nabla_U V + h(U, V) \), and \( \tilde{\nabla}_U \xi = -A_{\xi}U + \nabla_U \xi \), respectively for each \( U, V \in \mathfrak{X}(M) \) and for the normal vector field \( \xi \) of \( M \), where \( h \) and \( A_{\xi} \) are denoted as the second fundamental form and shape operator. They are related as \( g(h(U, V), N) = g(A_N U, V) \). Now, for any \( U \in \mathfrak{X}(M) \) and for the normal vector field \( \xi \) of \( M \), we have:
\( (i) \) \( JU = PU + FU \), \( (ii) \) \( J\xi = t\xi + f\xi \),
where \( PU(t\xi) \) and \( FU(f\xi) \) are tangential to \( M \) and normal to \( M \), respectively. Similarly, the equations of Gauss are given by:
\[ R(U, V, Z, W) = \tilde{R}(U, V, Z, W) + g(h(U, W), h(V, Z)) - g(h(U, Z), h(V, W)). \]
for all \( U, V, Z, W \) are tangent \( M \), where \( R \) and \( \tilde{R} \) are defined as the curvature tensor of \( \tilde{M}^m \) and \( M^n \), respectively.

The mean curvature \( H \) of Riemannian submanifold \( M^n \) is given by
\[ H = \frac{1}{n} \text{trace}(h). \]

A submanifold \( M^n \) of Riemannian manifold \( \tilde{M}^m \) is said to be totally umbilical and totally geodesic if \( h(U, V) = g(U, V)H \) and \( h(U, V) = 0 \), for any \( U, V \in \mathfrak{X}(M) \), respectively, where \( H \) is the mean curvature vector of \( M^n \). Furthermore, if \( H = 0 \), then \( M^n \) is minimal in \( \tilde{M}^m \).

A new class called a “pointwise slant submanifold” has been studied in almost Hermitian manifolds by Chen-Gray [35]. They provided the following definitions of these submanifolds:

**Definition 1.** [35] A submanifold \( M^n \) of an almost Hermitian manifold \( \tilde{M}^{2m} \) is a pointwise slant if, for any non-zero vector \( X \in \mathfrak{X}(T_xM) \) and each given point \( x \in M^n \), the angle \( \theta(X) \) between \( JX \) and tangent space \( T_xM \) is free from the choice of the nonzero vector \( X \). In this case, the Wirtinger angle become a real-valued function and it is non-constant along \( M^n \), which is defined on \( T^*M \) such that \( \theta : T^*M \rightarrow \mathbb{R} \).

Chen-Gray in [35] derived a characterization for the pointwise slant submanifold, where \( M^n \) is a pointwise slant submanifold if, and only if, there exists a constant \( \lambda \in [0, 1] \) such that \( P^2 = -\cos^2 \theta I \), where \( P \) is a \((1,1)\) tensor field and \( I \) is an identity map. For more classifications, we referred to [35].

Following the above concept, a pointwise bi-slant immersion was defined by Chen-Uddin in [18], where they defined it as follows:

**Definition 2.** A submanifold \( M^n \) of an almost Hermitian manifold \( \tilde{M}^{2m} \) is said to be a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions \( D_{\theta_1} \) and \( D_{\theta_2} \), such that:
\( (i) \) \( TM^n = D_{\theta_1} \oplus D_{\theta_2} \);
\( (ii) \) \( J D_{\theta_1} \perp D_{\theta_2} \) and \( J D_{\theta_2} \perp D_{\theta_1} \);
Each distribution $D_{\theta_i}$ is a pointwise slant with a slant function $\theta_i : T^i M \to \mathbb{R}$ for $i = 1, 2$.

**Remark 1.** A pointwise bi-slant submanifold is a bi-slant submanifold if each slant functions $\theta_i : T^i M \to \mathbb{R}$ for $i = 1, 2$ are constant along $M^n$ (see [13]).

**Remark 2.** If $\theta_1 = \alpha \theta_2$ or $\theta_2 = \alpha \theta_1$, then $M^n$ is called a pointwise pseudo-slant submanifold (see [33]).

**Remark 3.** If $\theta_1 = 0$ or $\theta_2 = 0$, in this case, $M^n$ is a coinciding pointwise semi-slant submanifold (see [14,34]).

**Remark 4.** If $\theta_2 = \alpha \theta_1$ and $\theta_1 = 0$, then $M^n$ is CR-submanifold of the almost Hermitian manifold.

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\tilde{M}^m$, and denoted at $\tilde{\tau}(T_x \tilde{M}^m)$, which, at some $x$ in $\tilde{M}^m$, is given:

$$\tilde{\tau}(T_x \tilde{M}^m) = \sum_{1 \leq \alpha \beta \leq m} \tilde{K}_{\alpha\beta}, \quad (7)$$

where $\tilde{K}_{\alpha\beta} = \tilde{K}(e_\alpha \wedge e_\beta)$. It is clear that the first equality (7) is congruent to the following equation, which will be frequently used in subsequent proof:

$$2\tilde{\tau}(T_x \tilde{M}^m) = \sum_{1 \leq \alpha \beta \leq m} \tilde{K}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n. \quad (8)$$

Similarly, scalar curvature $\tilde{\tau}(L_x)$ of $L$-plan is given by:

$$\tilde{\tau}(L_x) = \sum_{1 \leq \alpha \beta \leq m} \tilde{K}_{\alpha\beta}, \quad (9)$$

An orthonormal basis of the tangent space $T_x M$ is $\{e_1, \ldots, e_n\}$ such that $e_r = (e_{n+1}, \ldots, e_m)$ belong to the normal space $T^\perp M$. Then, we have:

$$h'_{\alpha\beta} = g(h(e_\alpha, e_\beta), e_r),$$

$$||h||^2 = \sum_{\alpha, \beta = 1}^n g(h(e_\alpha, e_\beta), h(e_\alpha, e_\beta)). \quad (10)$$

Let $K_{\alpha\beta}$ and $\tilde{K}_{\alpha\beta}$ be the sectional curvatures of the plane section spanned by $e_\alpha$ and $e_\beta$ at $x$ in a submanifold $M^n$ and a Riemannian manifold $\tilde{M}^m$, respectively. Thus, $K_{\alpha\beta}$ and $\tilde{K}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_\alpha, e_\beta\}$ at $x$. Thus, from the Gauss Equation (6)(i), we have:

$$2\tau(T_x M^n) = K_{\alpha\beta} = 2\tilde{\tau}(T_x M^m) + \sum_{r = n+1}^m \left(h'_{\alpha\beta} h'_r - (h'_r)^2\right)$$

$$= \tilde{K}_{\alpha\beta} + \sum_{r = n+1}^m \left(h'_{\alpha\beta} h'_r - (h'_r)^2\right). \quad (11)$$

The following consequences come from (6) and (11), as:

$$\tau(T_x M^n_1) = \sum_{r = n+1}^m \sum_{i \leq j \leq m} \left(h''_{ij} h''_{ij} - (h''_{ij})^2\right) + \tilde{\tau}(T_x M^m_1). \quad (12)$$
Similarly, we have:
\[ \tau(T_\alpha M_{n1}^{n2}) = \sum_{r=1}^{n} \sum_{i=1}^{n} \left( H'_\alpha h'_{bb} - (h'_{ab})^2 \right) + \tau(T_\alpha M_{n2}^{n2}). \]  
\[ \text{(13)} \]

Assume that \( M_{n1}^{n1} \) and \( M_{n2}^{n2} \) are two Riemannian manifolds with their Riemannian metrics \( g_1 \) and \( g_2 \), respectively. Let \( f \) be a smooth function defined on \( M_{n1}^{n1} \). Then, the warped product manifold \( M^n = M_{n1}^{n1} \times_f M_{n2}^{n2} \) is the manifold \( M_{n1}^{n1} \times M_{n2}^{n2} \) furnished by the Riemannian metric \( g = g_1 + f^2 g_2 \), which defined in [36]. When considering that the \( M^n = M_{n1}^{n1} \times_f M_{n2}^{n2} \) is the warped product manifold, then for any \( X \in \mathfrak{X}(M_1) \) and \( Z \in \mathfrak{X}(M_2) \), we find that:
\[ \nabla_\alpha X = \nabla_\beta Z = (X \ln f) Z. \]  
\[ \text{(14)} \]

Let \( \{e_1, \cdots, e_n\} \) be an orthonormal frame for \( M^n \); then, summing up the vector fields such that:
\[ \sum_{i=1}^{n_1} \sum_{\beta=1}^{n_2} K(e_\alpha \wedge e_\beta) = \sum_{i=1}^{n_1} \sum_{\beta=1}^{n_2} \left( (\nabla_{e_\alpha} e_\beta) \ln f - e_\alpha (e_\beta \ln f) - (e_\alpha \ln f)^2 \right). \]

From (Equation (3.3) in [11]), the above equation implies that:
\[ \sum_{i=1}^{n_1} \sum_{\beta=1}^{n_2} K(e_\alpha \wedge e_\beta) = n_2 \left( \Delta (\ln f) - ||\nabla (\ln f)||^2 \right) = \frac{n_2 \Delta f}{f}. \]  
\[ \text{(15)} \]

Remark 5. A warped product manifold \( M^n = M_{n1}^{n1} \times_f M_{n2}^{n2} \) is said to be trivial or a simple Riemannian product manifold if the warping function \( f \) is constant.

3. Main Inequality for Warped Product Pointwise Bi-Slant Submanifolds

To obtain similar inequalities like Theorem 1, for warped product pointwise bi-slant submanifolds of complex space forms, we need to recall the following lemma.

Lemma 1. [10] Let \( a_1, a_2, \ldots, a_n, a_{n+1} \) be \( n + 1 \) be real numbers with
\[ \left( \sum_{i=1}^{n} a_i \right)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a_{n+1}), n \geq 2. \]

Then \( 2a_1 a_2 \geq a_3 \) holds if and only if \( a_1 + a_2 = a_3 = \cdots = a_k. \)

Proof of Theorem 2. If substitute \( X = Z = e_\alpha \) and \( Y = W = e_\beta \) for \( 1 \leq \alpha, \beta \leq n \) in (4), and (6), taking summing up then
\[ \sum_{a, \beta=1}^{n} \tilde{R}(e_\alpha, e_\beta, e_\alpha, e_\beta) = \frac{C}{4} \left( n(n-1) + 3 \sum_{a, \beta=1}^{n} g^2(Je_\alpha, e_\beta) \right). \]  
\[ \text{(16)} \]

As \( M^n \) is a pointwise bi-slant submanifold, we defined an adapted orthonormal frame as \( n = 2d_1 + 2d_2 \) follows \( \{e_1, e_2 = \sec \theta_1 P e_1, \cdots, e_{2d_1-1}, e_{2d_1} = \sec \theta_1 P e_{2d_1-1}, \cdots, e_{2d_1+1}, e_{2d_1+2} = \sec \theta_2 P e_{2d_1+1}, \cdots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec \theta_2 P e_{2d_1+2d_2-1} \} \). Thus, we defined it such that \( g(e_1, J e_2) = -g(J e_1, e_2) = g(e_1, e_1, \sec \theta_1 P e_1), \) which implies that \( g(e_1, J e_2) = -\sec \theta_1 g(P e_1, P e_1). \)
Following ((2.8) in [32]), we get \(g(e_1, J e_2) = \cos \theta_1 g(e_1, e_2)\). Therefore, we easily obtained the following relation:

\[
g^2(e_\alpha, J e_\beta) = \begin{cases} 
\cos^2 \theta_1, & \text{for each } \alpha = 1, \ldots, 2d_1 - 1, \\
\cos^2 \theta_2, & \text{for each } \beta = 2d_1 + 1, \ldots, 2d_1 + 2d_1 - 1.
\end{cases}
\]

Hence, we have:

\[
\sum_{\alpha, \beta = 1}^n g^2(J e_\alpha, e_\beta) = (n_1 \cos^2 \theta + n_2 \cos^2 \phi).
\]

(17)

Following from (17), (16), and (6), we find that:

\[
2^r = \frac{c}{4} n(n - 1) + \frac{c}{4} \left(3n_1 \cos^2 \theta_1 + 3n_2 \cos^2 \theta_2\right) + n^2||H||^2 - ||h||^2.
\]

(18)

Let us assume that:

\[
\delta = 2^r - \frac{c}{4} n(n - 1) - \frac{c}{4} \left(3n_1 \cos^2 \theta_1 + 3n_2 \cos^2 \theta_2\right) - \frac{n^2}{2}||H||^2.
\]

(19)

Then, from (19), and (18), we get:

\[
n^2||H||^2 = 2(\delta + ||h||^2).
\]

(20)

Thus, from an orthogonal frame \(\{e_1, e_2, \ldots, e_n\}\), the proceeding equation takes the new form:

\[
\left(\sum_{r=n+1}^{2n} \sum_{i=1}^n h_{AA}^r\right)^2 = 2\left(\delta + \sum_{r=n+1}^{2n} \sum_{i=1}^n (h_{AA}^r)^2 + \sum_{r=n+1}^{2n+1} \sum_{i=1}^n (h_{AB}^r)^2\right)
\]

(21)

This can be expressed in more detail, such as:

\[
\frac{1}{2} \left(\sum_{A=2}^{n_1} h_{11}^{n_1} + \sum_{A=2}^{n_1} h_{AA}^{n_1} + \sum_{l=n_1+1}^{n} h_{ll}^{n_1+1}\right)^2 = \delta + \left(h_{11}^{n_1+1}\right)^2 + \sum_{A=2}^{n_1} \left(h_{AA}^{n_1+1}\right)^2 + \sum_{l=n_1+1}^{n} \left(h_{ll}^{n_1+1}\right)^2
\]

\[
- \sum_{2 \leq B \neq q \leq n_1} h_{BB}^{n_1+1} h_{qq}^{n_1+1} - \sum_{n_1+1 \leq l \neq s \leq n} h_{ll}^{n_1+1} h_{ss}^{n_1+1}
\]

\[
+ \sum_{A<B=1}^n (h_{AB}^{n_1+1})^2 + \sum_{r=n+1}^{2n} \sum_{A,B=1}^n (h_{AB}^r)^2.
\]

(22)

Assume that \(a_1 = h_{11}^{n_1+1}, a_2 = \sum_{A=2}^{n_1} h_{AA}^{n_1+1}\), and \(a_3 = \sum_{l=n_1+1}^{n} h_{ll}^{n_1+1}\). Then, applying Lemma 1 in (22), we derive:

\[
\frac{\delta}{2} + \sum_{A<B=1}^n \left(h_{AB}^{n_1+1}\right)^2 + \frac{1}{2} \sum_{r=n+1}^{2n} \sum_{A,B=1}^n (h_{AB}^r)^2 \leq \sum_{2 \leq B \neq q \leq n_1} h_{BB}^{n_1+1} h_{qq}^{n_1+1}
\]

\[
+ \sum_{n_1+1 \leq l \neq s \leq n} h_{ll}^{n_1+1} h_{ss}^{n_1+1}.
\]

(23)
with equality holds in (23) if and only if
\[ \sum_{A=2}^{n_1} h'_{AA} = \sum_{B=n_1+1}^{n} h'_{BB}. \] (24)

On the other hand, from (15), we have:
\[ \frac{n_2 \Delta f}{f} = \tau - \sum_{1 \leq A < B \leq n} K(e_A \wedge e_B) - \sum_{n_1+1 \leq l < q \leq n} K(e_l \wedge e_q). \] (25)

Then from (6) and the scalar curvature for the complex space form (11), we get:
\[ \frac{n_2 \Delta f}{f} = \tau - \frac{n_1(n_1-1)c}{8} - \frac{3n_1c}{4} \cos^2 \theta_1 - \frac{2m}{r=n+1} \sum_{1 \leq A < B \leq n} (h'_{AA}h'_{BB} - (h'_{AB})^2) \]
\[ - \frac{n_2(n_2-1)c}{8} - \frac{3n_2c}{4} \cos^2 \theta_2 - \frac{2m}{r=n+1} \sum_{n_1+1 \leq l < q \leq n} (h'_{ll}h'_{qq} - (h'_{lq})^2). \] (26)

Now from (23) and (26), we have:
\[ \frac{n_2 \Delta f}{f} \leq \rho - \frac{n(n-1)c}{8} + \frac{n_1n_2c}{4} - \frac{3n_1c}{4} \cos^2 \theta_1 - \frac{\delta}{2} - \frac{3n_2c}{4} \cos^2 \theta_2. \] (27)

Using (19) in the above equation and relation \( \Delta f = \Delta(\ln f) - ||\nabla \ln f||^2 \), we derive:
\[ n_2 \left( \Delta(\ln f) - ||\nabla \ln f||^2 \right) \leq \frac{n^2}{4} ||H||^2 + \frac{c}{4} \left( n_1n_2 + 3n_1 \cos^2 \theta_1 + 3n_2 \cos^2 \theta_2 \right). \] (28)

which implies inequality. The equality sign holds in (2) if, and only if, the leaving terms in (23) and (24) imply that:
\[ \sum_{r=n+2}^{2m} \sum_{B=1}^{n_1} h_{BB} = \sum_{r=n+2}^{2m} \sum_{A=n_1+1}^{n_1} h'_{AA} = 0, \] (29)

and \( n_1H_1 = n_2H_2 \), where \( H_1 \) and \( H_2 \) are partially mean curvature vectors on \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively. Moreover, also from (23), we find that
\[ h'_{AB} = 0, \text{ for each } 1 \leq A \leq n_1 \]
\[ n_1 + 1 \leq B \leq n \]
\[ n_1 + 1 \leq r \leq 2m. \] (30)

This shows that \( \varphi \) is a mixed, totally geodesic immersion. The converse part of (30) is true in a warped product pointwise bi-slant into the complex space form. Thus, we reached our promised result.

Consequences of Theorem 2

Inspired by the research in [6,34] and using the Remark 3 in Theorem 2 for pointwise semi-slant warped product submanifolds, we obtained:
Corollary 1. Let \( \varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \rightarrow \tilde{M}^{2m}(c) \) be an isometric immersion from the warped product pointwise semi-slant submanifold into a complex space form \( \tilde{M}^{2m}(c) \), where \( M_1^{n_1} \) is the holomorphic and \( M_2^{n_2} \) is the pointwise slant submanifolds of \( \tilde{M}^{2m}(c) \). Then, we have the following inequality:

\[
\Delta(\ln f) \leq ||\nabla \ln f||^2 + \frac{n_2}{4n_2} ||H||^2 + \frac{n_1c}{4} \left( n_1 + n_2 \cos^2 \theta \right),
\]

where \( n_i = \dim M_i, i = 1,2 \). Furthermore, \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operator on \( M_1^{n_1} \), respectively, and \( H \) is the mean curvature vector of \( M^n \). The equality sign holds in (31) if, and only if, \( n_1H_1 = n_2H_2 \), where \( H_1 \) and \( H_2 \) are the mean curvature vectors along \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively, and \( \varphi \) is a mixed, totally geodesic immersion.

From the motivation studied in [14,34], we present the following consequence of Theorem 2 by using the Remark 2 for a nontrivial warped product pointwise pseudo-slant submanifold of a complex space, such that:

Corollary 2. Let \( \varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \rightarrow \tilde{M}^{2m}(c) \) be an isometric immersion from the warped product pointwise pseudo-slant submanifold into a complex space form \( \tilde{M}^{2m}(c) \), such that \( M_1^{n_1} \) is a totally real and \( M_2^{n_2} \) is a pointwise slant submanifold of \( \tilde{M}^{2m}(c) \). Then, we have the following inequality:

\[
\Delta(\ln f) \leq ||\nabla \ln f||^2 + \frac{n_2}{4n_2} ||H||^2 + \frac{n_1c}{4} \left( n_1 + n_2 \cos^2 \theta \right),
\]

where \( n_i = \dim M_i, i = 1,2 \). Furthermore, \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operator on \( M_1^{n_1} \), respectively, and \( H \) is the mean curvature vector of \( M^n \). The equality condition holds in (32) if, and only if, the following satisfies

\[
\frac{H_1}{H_2} = \frac{n_2}{n_1},
\]

where \( H_1 \) and \( H_2 \) are the mean curvature vectors along \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively, and \( \varphi \) is a mixed, totally geodesic isometric immersion.

Corollary 3. Let \( \varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \rightarrow \tilde{M}^{2m}(c) \) be an isometric immersion from the warped product pointwise pseudo-slant submanifold into a complex space form \( \tilde{M}^{2m}(c) \), such that \( M_1^{n_1} \) is a pointwise slant and \( M_2^{n_2} \) is a totally real submanifold of \( \tilde{M}^{2m}(c) \). Then, we have the following:

\[
\Delta(\ln f) \leq ||\nabla \ln f||^2 + \frac{n_2}{4n_2} ||H||^2 + \frac{n_1c}{4} \left( n_1 + n_2 \cos^2 \theta \right),
\]

where \( n_i = \dim M_i, i = 1,2 \). Furthermore, \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operator on \( M_1^{n_1} \), respectively, and \( H \) is the mean curvature vector of \( M^n \). This equality holds in (33) if, and only if, \( \varphi \) is a mixed, totally geodesic isometric immersion and the following satisfies

\[
\frac{H_1}{H_2} = \frac{n_2}{n_1},
\]

where \( H_1 \) and \( H_2 \) are the mean curvature vectors along \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively.

Similarly, using Remark 4 and from [17], we got the following result from Theorem 2:
Corollary 4. Let \( \varphi : M^n = M_1^{m_1} \times f M_2^{m_2} \rightarrow \tilde{M}^{2m}(c) \) be an isometric immersion from a CR-warped product into a complex space form \( \tilde{M}^{2m}(c) \), such that \( M_1^{m_1} \) is a holomorphic submanifold and \( M_2^{m_2} \) is a totally real submanifold of \( \tilde{M}^{2m}(c) \). Then, we get the following:

\[
\Delta(\ln f) \leq \||\nabla \ln f||^2 + \frac{n_2^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3n_1 c}{4n_2},
\]

(34)

where \( n_i = \dim M_i, \ i = 1, 2 \). Furthermore, \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operator on \( M_i^{m_i} \), respectively, and \( H \) is the mean curvature vector of \( M_i^{m_i} \). The same holds in (34) if, and only if, \( \varphi \) is mixed and totally geodesic, and \( n_1 H_1 = n_2 H_2 \), where \( H_1 \) and \( H_2 \) are the mean curvature vectors on \( M_1^{m_1} \) and \( M_2^{m_2} \), respectively.

In particular, if both pointwise slant functions \( \theta_1 = \theta_2 = \frac{\pi}{2} \), then \( M^n \) is becomes a totally real warped product submanifold—thus, we obtain:

Corollary 5. Let \( \varphi : M^n = M_1^{m_1} \times f M_2^{m_2} \rightarrow \tilde{M}^{2m}(c) \) be an isometric immersion from an \( n \)-dimensional, totally real warped product submanifold into a \( 2m \)-dimensional complex space form \( \tilde{M}^{2m}(c) \), where \( M_1^{m_1} \) and \( M_2^{m_2} \) are totally real submanifolds of \( \tilde{M}^{2m}(c) \). Then, we have the following:

\[
\Delta(\ln f) \leq \||\nabla \ln f||^2 + \frac{n_2^2}{4n_2}||H||^2 + \frac{n_1 c}{4},
\]

(35)

where \( n_i = \dim M_i, \ i = 1, 2 \) and \( \Delta \) is the Laplacian operator on \( M_1^{m_1} \). The same holds in (35) if, and only if, \( \varphi \) is mixed and totally geodesic, and the following satisfies

\[
\frac{H_1}{H_2} = \frac{n_2}{n_1},
\]

where \( H_1 \) and \( H_2 \) are the mean curvature vectors on \( M_1^{m_1} \) and \( M_2^{m_2} \), respectively.

Proof of Theorem 3. In this direction, we consider the warped product pointwise bi-slant submanifolds as a compact oriented Riemannian manifold without boundary. If the inequality (2) holds:

\[
\Delta(\ln f) - \||\nabla \ln f||^2 \leq \frac{n_2^2}{4n_2}||H||^2 + \frac{c}{4n_2} \left( n_1 n_2 - 3n_1 \cos^2 \theta_1 - 3n_2 \cos^2 \theta_2 \right).
\]

(36)

Since \( M^n \) is a compact oriented Riemannian submanifold without boundary, then we have following formula with respect to the volume element:

\[
\int_{M^n} \Delta f dV = 0.
\]

(37)

From the hypothesis of the theorem, \( M^n \) is a compact warped product submanifold; then from (37), we derive:

\[
\int_{M} \left( \frac{c}{4n_2} \left( 3n_1 \cos^2 \theta_1 + 3n_2 \cos^2 \theta_2 - n_1 n_2 \right) - \frac{1}{4n_2} \sum_{i=1}^{n} (h^n_{i})^2 \right) dV \leq \int_{M} (||\nabla \ln f||^2) dV.
\]

(38)

Now, we assume that \( M^n \) is a Riemannian product, and the warping function \( f \) must be constant on \( M^n \). Then, from (38), we get the inequality (3).

\[\square\]
Conversely, let the inequality (3) hold; then from (38), we derive:

\[ 0 \leq \int_{M^n} \left( |\nabla \ln f|^2 \right) \leq 0. \]

The above condition implies that \( |\nabla \ln f|^2 = 0 \), where this means that \( f \) is a constant function on \( M^n \). Hence, \( M^n \) is simply a Riemannian product of \( M_1^{n_1} \) and \( M_2^{n_2} \), respectively. Thus, the theorem is proved. We give some other important corollaries as consequences of Theorem 2, as follows:

**Corollary 6.** Let \( M^n = M_1^{n_1} \times_f M_2^{n_2} \) be a warped product pointwise bi-slant submanifold of a complex space form \( \tilde{M}^{2m}(c) \) with warping function \( f \), such that \( n_1 = \dim M_1 \) and \( n_2 = \dim M_2 \). If \( \varphi \) is an isometrically minimal immersion from warped product \( M^n \) into \( \tilde{M}^{2m}(c) \), then we obtain:

\[
\Delta(\ln f) \leq |\nabla \ln f|^2 + \frac{c}{4n_2} \left( n_1 n_2 - 3n_1 \cos^2 \theta_1 - 3n_2 \cos^2 \theta_2 \right). \tag{39}
\]

**Corollary 7.** Let \( M^n = M_1^{n_1} \times_f M_2^{n_2} \) be a warped product pointwise bi-slant submanifold of a complex space form \( \tilde{M}^{2m}(c) \) with warping function \( f \), such that \( n_1 = \dim M_1 \) and \( n_2 = \dim M_0 \). Then, there is no existing minimal isometric immersion \( \varphi \) from warped product \( M^n \) into \( \tilde{M}^{2m}(c) \) with:

\[
\Delta(\ln f) > |\nabla \ln f|^2 + \frac{c}{4n_2} \left( n_1 n_2 - 3n_1 \cos^2 \theta_1 - 3n_2 \cos^2 \theta_2 \right). \tag{40}
\]

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