On Sliced Spaces: Global Hyperbolicity Revisited

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Abstract: We give a topological condition for a generic sliced space to be globally hyperbolic without any hypothesis on lapse function, shift function, and spatial metric.

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1. Preliminaries

The definition of a sliced space, which one can read in Reference [1], is a continuation of a study in References [2] and [3] on systems of Einstein equations.

Let \( V = M \times I \), where \( M \) is an \( n \)-dimensional smooth manifold, and \( I \) is an interval of the real line, \( \mathbb{R} \). We equip \( V \) with a \((n+1)\)-dimensional Lorentz metric \( g \), which splits in the following way:

\[
g = -N^2(\theta^0)^2 + g_{ij} \theta^i \theta^j,
\]

where \( \theta^0 = dt, \theta^i = dx^i + \beta^i dt, N = N(t, x^i) \) is the lapse function, \( \beta^i(t, x^i) \) is the shift function and \( M_t = M \times \{t\} \), spatial slices of \( V \), are spacelike submanifolds equipped with the time-dependent spatial metric \( g_t = g_{ij} dx^i dx^j \). Such product space \( V \) is called a sliced space.

Throughout the paper, we consider \( I = \mathbb{R} \).

The author in Reference [1] considered sliced spaces with uniformly bounded lapse, shift, and spatial metric; by this hypothesis, it is ensured that parameter \( t \) measures up to a positive factor bounded (below and above) the time along the normals to spacelike slices \( M_t \), the \( g_t \) norm of the shift vector \( \beta \) is uniformly bounded by a number, and the time-dependent metric \( g_{ij} dx^i dx^j \) is uniformly bounded (below and above) for all \( t \in I = \mathbb{R} \), respectively.

Given the above hypothesis, in the same article, the following theorem was proved.

**Theorem 1** (Cotsakis). Let \((V, g)\) be a sliced space with uniformly bounded lapse \( N \), shift \( \beta \) and spatial metric \( g_t \). Then, the following are equivalent:

1. \((M_0, \gamma)\) a complete Riemannian manifold.
2. Spacetime \((V, g)\) is globally hyperbolic.

In this article, we review global hyperbolicity of sliced spaces in terms of the product topology defined on space \( M \times \mathbb{R} \) for some finite dimensional smooth manifold \( M \).

2. Strong Causality of Sliced Spaces

Let \((V = M \times \mathbb{R}, g)\) be a sliced space. Consider product topology \( T_P \) on \( V \). Since \( M \) is finite-dimensional, a base for \( T_P \) consists of all sets of form \( A \times B \), where \( A \in T_M \) and \( B \in T_{\mathbb{R}} \).
Here, $T_M$ denotes the natural topology of manifold $M$ where, for an appropriate Riemann metric $h$, it has a base consisting of open balls $B^h_b(x)$, and $T_\mathbb{R}$ is the usual topology on the real line, with a base consisting of open intervals $(a,b)$. For trivial topological reasons, we can restrict our discussion on $T_p$ to basic-open sets $B^h_b(x) \times (a,b)$, which can intuitively be called “open cylinders” in $V$.

We remind that the Alexandrov topology $T_A$ (see Reference [4]) has a base consisting of open sets of the form $< x, y > = I^+(x) \cap I^-(y)$, where $I^+(x) = \{ z \in V : x \ll z \}$ and $I^-(y) = \{ z \in V : z \ll y \}$, where $\ll$ is the chronological order defined as $x \ll y$ if there exists a future-oriented timelike curve joining $x$ with $y$. By $J^+(x)$, one denotes the topological closure of $I^+(x)$, and by $J^-(y)$ that one of $I^-(y)$.

We use the definition of global hyperbolicity from Reference [4], where one can read about global causality conditions in more detail, as well as characterizations for strong causality. In particular, a spacetime is strongly causal iff it possesses no closed timelike curves, and global hyperbolicity is an important causal condition in a spacetime related to major problems such as spacetime singularities and cosmic censorship.

**Definition 1.** A spacetime is globally hyperbolic iff it is strongly causal and the “causal diamonds” $J^+(x) \cap J^-(y)$ are compact.

We prove the following theorem:

**Theorem 2.** Let $(V, g)$ be a Hausdorff sliced space. Then, the following are equivalent.

1. $V$ is strongly causal.
2. $T_A \equiv T_p$.
3. $T_A$ is Hausdorff.

**Proof.** Here, 2. implies 3. is obvious and that 3. implies 1. can be found in Reference [4].

For 1. implies 2., we consider two events $X, Y \in V$, such that $X \neq Y$; we note that each $x \in V$ has two coordinates, say $(x_1, x_2)$, where $x_1 \in M$ and $x_2 \in \mathbb{R}$. Obviously, $X \in M_x = M \times \{ x \}$ and $Y \in M_y = M \times \{ y \}$. Then, $< X, Y > = I^+(X) \cap I^-(Y) \in T_A$. Let also $A \in M = M \times \{ a \}$, where $a < x (< \in$ is the natural order on $\mathbb{R}$) and $B \in M_b = M \times \{ b \}$, where $y < b$. Consider some $\epsilon > 0$, such that $B^h_b(A) \in M$. Obviously, $B^h_b(A) \times (a, b) \in T_p$ and, for $\epsilon > 0$ sufficiently large enough, $< X, Y > \in B^h_b(A) \times (a, b)$. Thus, $< X, Y > \in T_p$.

For 2. implies 1., we consider $\epsilon > 0$, such that $B^h_b(A) \in T_M$, so that $B^h_b(A) \times (a, b) = B \in T_p$. We let strong causality hold at an event $P$ and consider $P \in B \in T_p$. We show that there exists $< X, Y > \in T_A$, such that $P < X, Y > \subset B$. Now, consider a simple region $R$ in $< X, Y >$ which contains $P$ and $P \in Q$, where $Q$ is a causally convex-open subset of $R$. Thus, we have $U, V \in Q$, such that $P < U, V > \subset Q$. Finally, $P < U, V > \subset Q \subset B$, and this completes the proof.

**3. Global Hyperbolicity of Sliced Spaces, Revisited**

For the following theorem, we use Nash’s result that refers to finite-dimensional manifolds (see Reference [5]).

**Theorem 3.** Let $(V, g)$ be a Hausdorff sliced space, where $V = M \times \mathbb{R}$, $M$ is an $n$-dimensional manifold and $g$ the $n + 1$ Lorentz metric in $V$. Then, $(V, g)$ is globally hyperbolic iff $T_p = T_A$, in $V$.

**Proof.** Given the proof of Theorem 2, strong causality in $V$ holds iff $T_p = T_A$ and, given Nash’s theorem, the closure of $B^h_b(x) \times (a, b)$ is compact.

We note that neither in Theorem 2 nor in Theorem 3 did we make any hypothesis on the lapse function, shift function, or spatial metric.
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