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A Note on the Sequence Related to Catalan Numbers

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Abstract: The main purpose of this paper is to find explicit expressions for two sequences and to solve two related conjectures arising from the recent study of sums of finite products of Catalan numbers by Zhang and Chen.

Keywords: new sequence; Catalan numbers; elementary and combinatorial methods; congruence; conjecture

MSC: 11B83; 11B75

1. Introduction

Let n be any non-negative integer. Then, $C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$ $(n = 0, 1, 2, 3, \cdots)$ are defined as the Catalan numbers. For example, the first several values of the Catalan numbers are $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$, $\cdots$. The generating function of the sequence $\{C_n\}$ is:

$$\frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} \cdot x^n = \sum_{n=0}^{\infty} C_n \cdot x^n. \tag{1}$$

This sequence occupies a pivotal position in combinatorial mathematics, so lots of counting problems are closely related to it. A great number of examples can be found in a study by Stanley [1]. Because of these, plenty of scholars have researched the properties of Catalan numbers and obtained a large number of vital and meaningful results. Interested readers can refer to the relevant references [2–26], which is not an exhaustive list. Very recently, Zhang and Chen [27] researched the calculation problem of the following convolution sums:

$$\sum_{a_1+a_2+\cdots+a_h=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_h}, \tag{2}$$

where the summation has taken over all h-dimension non-negative integer coordinates $(a_1, a_2, \cdots, a_h)$, such that the equation $a_1 + a_2 + \cdots + a_h = n$.

They first introduced two new recursive sequences, $C(h, i)$ and $D(h, i)$, and after the elementary and combinatorial methods, they proved the following two significant conclusions:

Theorem 1. *For any positive integer h, one gets the identity:*

$$\begin{aligned} &\sum_{a_1+a_2+\cdots+a_{2h+1}=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_{2h+1}} \\ &= \frac{1}{(2h)!} \sum_{i=0}^{h} C(h,i) \sum_{j=0}^{\min(n,i)} \frac{(n-j+h+i)! \cdot C_{n-j+h+i}}{(n-j)!} \cdot \binom{i}{j} \cdot (-4)^j, \end{aligned}$$

where the sequence $C(h,i)$ is defined as $C(1,0) = -2$, $C(h,h) = 1$, $C(h+1,h) = C(h,h-1) - (8h+2) \cdot C(h,h)$, $C(h+1,0) = 8 \cdot C(h,1) - 2 \cdot C(h,0)$, and for all integers $1 \le i \le h-1$, we acquire the recursive formula:

$$C(h+1,i) = C(h,i-1) - (8i+2) \cdot C(h,i) + (4i+4)(4i+2) \cdot C(h,i+1).$$

Theorem 2. *For any positive integer h and non-negative n, one can obtain:*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_{2h}=n} C_{a_1} \cdot C_{a_2} \cdot C_{a_3} \cdots C_{a_{2h}} \\ = &\frac{1}{(2h-1)!} \sum_{i=0}^{h-1} \sum_{j=0}^{n} D(h,i+1) \cdot \binom{i+\frac{1}{2}}{j} \cdot (-4)^j \cdot \frac{(n-j+h+i)! \cdot C_{n-j+h+i}}{(n-j)!},\end{aligned}$$

where $\binom{n+\frac{1}{2}}{i} = \left(n+\frac{1}{2}\right) \cdot \left(n-1+\frac{1}{2}\right) \cdots \left(n-i+1+\frac{1}{2}\right)/i!$, the sequence $D(k,i)$ are defined as $D(k,0) = 0$, $D(k,k) = 1$, $D(k+1,k) = D(k,k-1) - (8k-2)$, $D(k+1,1) = 24D(k,2) - 6D(k,1)$, and for all integers $1 \le i \le k-1$,

$$D(k+1,i) = D(k,i-1) - (8i-2) \cdot D(k,i) + 4i(4i+2) \cdot D(k,i+1).$$

Meanwhile, through numerical observation, Zhang and Chen [27] also proposed the following two conjectures:

Conjecture 1. *Let p be a prime. Then, for any integer $0 \le i < \frac{p+1}{2}$, we obtain the congruence:*

$$C\left(\frac{p+1}{2}, i\right) \equiv 0 \bmod p(p+1).$$

Conjecture 2. *Let p be a prime. Then, for any integer $0 \le i < \frac{p+1}{2}$, we obtain the congruence:*

$$D\left(\frac{p+1}{2}, i\right) \equiv 0 \bmod p(p-1).$$

For easy comparison, here we list some of the values of $C(h,i)$ and $D(h,i)$ with $1 \le h \le 6$ and $0 \le i \le h$ in the following Tables 1 and 2.

Table 1. Values of $C(k,i)$.

$C(k,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$k=1$	−2	1					
$k=2$	12	−12	1				
$k=3$	−120	180	−30	1			
$k=4$	1680	−3360	840	−56	1		
$k=5$	−30,240	75,600	−25,200	2520	−90	1	
$k=6$	665,280	−1,995,840	831,600	−110,880	5940	−132	1

Table 2. Values of $D(k,i)$.

$D(k,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$k=1$	0	1					
$k=2$	0	−6	1				
$k=3$	0	60	−20	1			
$k=4$	0	−840	420	−42	1		
$k=5$	0	15,120	−10,080	1512	−72	1	
$k=6$	0	−332640	277,200	−55,440	3960	−110	1

Based on these two tables and a large number of numerical calculations, we found that these conjectures are not only correct, but also have generalized conclusions. Actually, they provide a simpler and clearer representation.

In this paper, by using some notes from Zhang and Chen's work [27] as well as some basic and combinatorial methods, we are going to prove the following:

Theorem 3. *Let h be a positive integer. Then, for any integer i with $0 \le i \le h$, we acquire the identity:*

$$C(h,i) = (-1)^{h-i} \cdot \frac{(2h)!}{(h-i)! \cdot (2i)!}.$$

Theorem 4. *Let h be a positive integer. Then, for any integer i with $1 \le i \le h$, we acquire the identity:*

$$D(h,i) = (-1)^{h-i} \cdot \frac{(2h-1)!}{(h-i)! \cdot (2i-1)!}.$$

Based on the above two theorems, we may instantly deduce the following two corollaries:

Corollary 1. *Let h be any positive integer. Then, for any integer $0 \le i \le h-1$, we gain the congruence:*

$$C\,(h,i) \equiv 0 \bmod 2h(2h-1).$$

Corollary 2. *Let h be any positive integer. Then, for any integer $0 \le i \le h-1$, we gain the congruence:*

$$D\,(h,i) \equiv 0 \bmod (2h-1)(2h-2).$$

Suppose that we consider p an odd prime, and that when $h = \frac{p+1}{2}$ in Corollary 1 and Corollary 2, combined with the identities $2h(2h-1) = p(p+1)$ and $(2h-1)(2h-2) = p(p-1)$, our Corollary 1 and Corollary 2 proves Conjecture 1 and Conjecture 2, respectively. Practically, they prove two more general conclusions.

Taking $n = 0$ in Theorem 1 and Theorem 2 and applying our theorems, we may instantly deduce the following two identities:

Corollary 3. *Let h be any positive integer. Then, we get the identity:*

$$\sum_{i=0}^{h} (-1)^{h-i} \binom{h+i}{2i} \cdot C_{h+i} = 1.$$

Corollary 4. *Let h be any positive integer. Then, we get the identity:*

$$\sum_{i=1}^{h} (-1)^{h-i} \binom{h+i-1}{2i-1} \cdot C_{h+i-1} = 1.$$

Some notes: If we replace $C(h,i)$ ($D(h,i)$) in Theorem 1 (Theorem 2) with the formula for $C(h,i)$ ($D(h,i)$) in our Theorem 3 (Theorem 4), then we can get a more accurate representation for convolution sums (2).

The proof of the results in this paper is uncomplicated, but guessing their specific forms is not easy.

2. Proofs of the Theorems

Actually, the recursive form of the sequence $C(h,i)$ or $D(h,i)$ is more complex, but as long as we are able to guess its accurate representation, it is not difficult to prove. First of all, combining the mathematical induction method, we are going to prove:

$$C(h,i) = (-1)^{h-i} \cdot \frac{(2h)!}{(h-i)! \cdot (2i)!}. \tag{3}$$

According to Table 1, we know that $C(1,0) = -2$, $C(1,1) = 1$, $C(2,0) = -12$, $C(2,1) = 12$, $C(2,2) = 1$, $C(3,0) = -120$, $C(3,1) = 180$, $C(3,2) = -30$, $C(3,3) = 1$. This means that (3) is correct for $h = 1,\ 2,\ 3$, and $0 \le i \le h$.

Assume that (3) is correct for integer $h = k$ and all $0 \le i \le k$. That is,

$$C(k,i) = (-1)^{k-i} \cdot \frac{(2k)!}{(k-i)! \cdot (2i)!},\ 0 \le i \le k. \tag{4}$$

Then, for $h = k+1$, if $i = h+1$, applying the definition of $C(h,i)$, we acquire $C(k+1,k+1) = 1$. If $i = 0$, combining the inductive hypothesis (4) and noting that $C(k+1,0) = 8C(k,1) - 2C(k,0)$, we obtain:

$$C(k+1,0) = 8 \cdot (-1)^{k-1} \cdot \frac{(2k)!}{(k-1)! \cdot 2!} - (-1)^k \cdot 2 \cdot \frac{(2k)!}{k!} = (-1)^{k+1} \frac{(2k+2)!}{(k+1)!}. \tag{5}$$

Suppose that $1 \le i \le k$. From (4) and the recursive properties of $C(h,i)$, we gain:

$$\begin{aligned} & C(k+1,i) = C(k,i-1) - (8i+2) \cdot C(k,i) + (4i+4)(4i+2) \cdot C(k,i+1) \\ & = (-1)^{k-i+1} \tfrac{(2k)!}{(k-i+1)!(2i-2)!} - (-1)^{k-i}(8i+2) \tfrac{(2k)!}{(k-i)!(2i)!} \\ & +(-1)^{k-i-1}(4i+4)(4i+2) \cdot \tfrac{(2k)!}{(k-i-1)!(2i+2)!} \\ & = (-1)^{k+1-i} \cdot \tfrac{(2k+2)!}{(k+1-i!) \cdot (2i)!}. \end{aligned} \tag{6}$$

According to (5) and (6), we know that the Formula (3) is correct for $h = k+1$ and all integers $0 \le i \le k+1$. Theorem 3 can then be proved by mathematical induction.

In a similar way, we can also prove Theorem 4 by mathematical induction. Since the proof process is the same as the proof of Theorem 3, it is omitted.

3. Conclusions

The main purpose of this paper was to give two specific expressions for the sequences $C(h,i)$ and $D(h,i)$. As for some applications of our results, we proved two conjectures proposed by Zhang and Chen in [27].

As a matter of fact, our results are more general and not subject to prime conditions. Meanwhile, using our formulae for $C(h,i)$ and $D(h,i)$ in the theorems, we can simplify the variety of results that appear in Reference [27].

This paper not only enriches the research content of the Catalan numbers, but can also be regarded as a supplement and further improvement to Zhang and Chen's work in [27].

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