On Ω Class of Mappings in a \( p \)-Cyclic Complete Metric Space

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Abstract: In this manuscript, we introduce the concept of Ω-class of self mappings on a metric space and a notion of \( p \)-cyclic complete metric space for a natural number \( p \geq 2 \). We not only give sufficient conditions for the existence of best proximity points for the Ω-class self-mappings that are defined on \( p \)-cyclic complete metric space, but also discuss the convergence of best proximity points for those mappings.

Keywords: \( p \)-cyclic maps; \( p \)-cyclic contractions; strict contractions; best proximity points

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1. Introduction

In the classical Banach fixed point theorem, the undertaking operator is necessarily continuous due to contraction inequality. This simple observation brings a natural question: Does a discontinuous contraction mapping possess a fixed point? The answer to this question is affirmative. Indeed, there are various approaches to overcome weakness of the discontinuous mapping for guaranteeing a fixed point. One of the significant results was constructed by Bryant [1] who proved the following result: In a complete metric space, if, for some positive integer \( n \geq 2 \), the \( n \)th iteration of the given mapping forms a contraction, then it possess a unique fixed point. Another outstanding approach was proposed by Kirk, Srinivasan and Veeramani [2] by introducing the notion of cyclic contraction. More precisely, every cyclic contraction in a complete metric space possess a unique fixed point. This statement is plain but significant when we compare with the results of Bryant. Attendantly, the concept of the cyclic contractions has been investigated densely by a considerable number of authors who bring several variants of the notion and derive a number of interesting results (see, e.g., [3–16] and the references therein).

Let there be a self-mapping on a metric space \((X, d)\). Suppose that \(A\) and \(B\) are non-empty subsets of \(X\) such that \(X = A \cup B\). A self-mapping \(T\) on \(A \cup B\) is called cyclic [2]

\[ T(A) \subset B \text{ and } T(B) \subset A. \]
Further, a mapping $T$ is called cyclic contraction [2] if there is a $k \in [0,1)$ such that the following inequality is satisfied:

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x \in X_i, y \in X_{i+1}, i \in \{1, 2, \ldots, m\}.$$ 

After this initial construction, several extension of cyclic mappings and cyclic contractions have been introduced. In this paper, we mainly follow the notations defined in [9].

2. Motivation

In [9], a notion of $p$-cyclic map is introduced. Let $B_1, B_2, \ldots, B_p \ (p \geq 2)$ be non-empty sets. A $p$-cyclic map $T : \bigcup_{i=1}^p B_i \rightarrow \bigcup_{i=1}^p B_i$ is defined such that $T(B_i) \subseteq B_{i+1}$, $\forall i \in \{1, 2, \ldots, p\}$. $x = x_0 \in B_i$ defines a sequence $(x_n) \subseteq \bigcup_{i=1}^p B_i$ as $x_n = T(x_{n-1})$. Then, $(x_{pn})$ is a subsequence in $B_i$, $(x_{pm+1})$ is a subsequence in $B_{i+1}$ and so on. The arrangement of such a sequence formed by a $p$-cyclic map motivated us to introduce a notion of $p$-cyclic sequence (Definition 6(1)). If $B_j$s are subsets of a metric space $(M, \rho)$, then we observe that, to obtain a best proximity point of $T$ under various contractive conditions (some of them given in the literature), it is enough to prove that: given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\rho(x_{pn}, x_{pn+1}) < \text{dist}(B_j, B_{i+1}) + \epsilon, \ \forall \ n, m \geq N_0.$$ 

This observation motivated us to introduce a concept of $p$-cyclic Cauchy sequence and $p$-cyclic complete metric space (Definition 6). In addition, while investigating the behavior of such $p$-cyclic maps, it is often the case that, if $\rho(x, y) > \text{dist}(B_i, B_{i+1})$, then $\rho(Tx, Ty) < \rho(x, y)$ and, if $\rho(x, y) = \text{dist}(B_j, B_{i+1})$, then $\rho(Tx, Ty) = \rho(x, y)$, $x \in B_i$, $y \in B_{i+1}$. This motivated us to call a $p$-cyclic map with this property as $p$-cyclic strict contraction map (Definition 7). Note that, if the distances between the adjacent sets are zero, then a $p$-cyclic strict contraction map is a strict contraction map in the usual sense. All such maps invariably satisfy the condition: $x, y \in B_i$, $\rho(T^{pn}x, T^{pn+1}y) \rightarrow \text{dist}(B_i, B_{i+1})$ as $n \rightarrow \infty$. In this paper, all $p$-cyclic maps which satisfy the above two properties are said to belong to class $\Omega$ (Definition 8). Finally, we prove the existence and convergence of best proximity points of $\Omega$ class of mappings in a $p$-cyclic complete metric space.

3. Preliminaries

In what follows, we give some definitions and fundamental results that are essential to understand and prove the main results.

**Definition 1** ([9], Definitions 3.1). For a non-empty set $M$, suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \ldots, B_p \ (p \geq 2)$ are non-empty subsets of $M$. Define $B_{p+i} := B_i$, for all $i \in \{1, 2, \ldots, p\}$. A map $T : \bigcup_{i=1}^p B_i \rightarrow \bigcup_{i=1}^p B_i$ is called a $p$-cyclic map if $T(B_i) \subseteq B_{i+1}$, $\forall i \in \{1, 2, \ldots, p\}$. If $p = 2$, the map $T$ is called cyclic. A point $x \in B_i$ is said to be a best proximity point of $T$ in $B_i$, if $\rho(x, Tx) = \text{dist}(B_i, B_{i+1})$, where $\text{dist}(B_i, B_{i+1}) := \inf \{\rho(x, y) : x \in B_i, y \in B_{i+1}\}$.

In this paper, we give the conditions for the underlying space and for the subsets of the space, to have a unique best proximity point under a $p$-cyclic map, if it exists, irrespective of the contraction condition imposed on the map.

**Proposition 1.** Let $B_1, B_2, \ldots, B_p \ (p \geq 2)$ be non-empty convex subsets of a strictly convex norm linear space $M$ such that $\text{dist}(B_i, B_{i+1}) > 0$, $i \in \{1, 2, \ldots, p\}$. Let $T : \bigcup_{i=1}^p B_i \rightarrow \bigcup_{i=1}^p B_i$ be a $p$-cyclic map. Then, $T$ has at most one best proximity point in $B_i$, $1 \leq i \leq p$. 

Proof. Let \( x, y \in B_i \) be such that \( \|x - Tx\| = \text{dist}(B_i, B_{i+1}) = \|y - Ty\|, 1 \leq i \leq p \). If \( x \neq y \), then
\[
\frac{\|x - Tx\|}{\text{dist}(B_i, B_{i+1})} = 1 \quad \text{and} \quad \frac{\|y - Ty\|}{\text{dist}(B_i, B_{i+1})} = 1.
\]
Since \( M \) is strictly convex, \( \|\frac{(x-Tx)+(y-Ty)}{2}\| < 1 \). Thus, we get
\[
\|\frac{x+y}{2} - (\frac{Ty+T(x+y)}{2})\| < \text{dist}(B_i, B_{i+1}),
\]
which is a contradiction. Hence, \( x = y \). \( \square \)

Let \( T \) be a \( p \)-cyclic map as given in Definition 1. \( T \) is said to be \( p \)-cyclic non expansive map if for all \( x \in B_i, y \in B_{i+1} \), the following holds:
\[
\rho(Tx, Ty) \leq \rho(x, y), \forall \ i \in \{1, 2, \ldots, p\}.
\]

The Lemma given below naturally follows for a \( p \)-cyclic non expansive map.

**Lemma 1** ([9], Lemma 3.3). For a non-empty set \( M \), suppose \( \rho : M \times M \rightarrow [0, \infty) \) forms a metric and \( B_1, B_2, \ldots, B_p \ (p \geq 2) \) are non-empty subsets of \( M \). If \( T : \bigcup_{i=1}^p B_i \rightarrow \bigcup_{i=1}^p B_i \) is a \( p \)-cyclic non-expansive map, then
\[
\text{dist}(B_i, B_{i+1}) = \text{dist}(B_{i+1}, B_{i+2}) = \text{dist}(B_i, B_2), \forall \ i \in \{1, 2, \ldots, p\}.
\] (1)

In addition, if \( \xi \in B_i \cap \mathcal{B}(T)_i \neq \emptyset \), then \( T^j\xi \in B_{i+j} \cap \mathcal{B}(T)_{i+j} \neq \emptyset \), for all \( j = 1, 2, \ldots, (p-1) \), where \( \mathcal{B}(T)_k \) is the set of best proximity point of the mapping \( T \) in \( B_k \).

In [5], the following lemma is proved, which is again proved here. This lemma is crucial to prove that a given sequence is Cauchy.

**Lemma 2** ([5], Lemma 3.7). For a uniformly convex Banach space \( (X, \|\cdot\|) \), we suppose that \( C, D \) are non-empty closed subsets of \( X \) and \( \{a_n\}, \{b_n\} \subset C \) and \( \{d_n\} \subset D \). If \( C \) is convex such that
\begin{enumerate}[(i)]
  \item \( \|a_n - d_n\| \rightarrow \text{dist}(C, D) \); and
  \item for every \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( m > n \geq N \), \( \|a_m - d_n\| \leq \text{dist}(C, D) + \epsilon \),
\end{enumerate}
then for all \( \epsilon > 0 \), there exists \( N_1 \in \mathbb{N} \) such that for all \( m > n \geq N_1 \), \( \|a_m - b_n\| \leq \epsilon \).

Proof. If \( \text{dist}(C, D) = 0 \), then, for a given \( \epsilon > 0 \), we can find positive integers \( N_2 \) and \( N_0 \) such that \( \|b_n - d_n\| \leq \frac{\epsilon}{2} \) for all \( n \geq N_2 \) and \( \|a_m - d_n\| \leq \frac{\epsilon}{2} \) for all \( m > n > N_0 \). Now, choosing \( N_1 = \max(N_0, N_2) \), then
\[
\|a_m - d_n\| \leq \|a_m - d_n\| + \|b_n - d_n\| \leq \epsilon, \text{ for all } m > n > N_1.
\]

Suppose \( \text{dist}(C, D) > 0 \); assuming the contrary, there exists \( \epsilon_0 > 0 \) such that, for every \( k \in \mathbb{N} \), there exists \( m_k > n_k \geq k \), for which \( \|a_{m_k} - d_{n_k}\| \geq \epsilon_0 \). Choose \( 0 < \gamma < 1 \) such that \( \epsilon_0 / \gamma > \text{dist}(C, D) \) and choose \( \epsilon \) such that \( 0 < \epsilon \leq \min \left( \frac{\epsilon_0}{\gamma} - \text{dist}(C, D), \frac{\text{dist}(C, D)\delta(\gamma)}{1 - \delta(\gamma)} \right) \), where \( \delta(\gamma) \) is the modulus of convexity. For this \( \epsilon > 0 \), there exists \( N_0 \) such that for all \( m_k > n_k \geq N_0 \), \( \|a_{m_k} - d_{n_k}\| < \text{dist}(C, D) + \epsilon \). In addition, there exists \( N_2 \) such that \( \|b_{n_k} - d_{n_k}\| < \text{dist}(C, D) + \epsilon \) for all \( n_k \geq N_2 \). Choose \( N_1 = \max(N_0, N_2) \).

By uniform convexity, for all \( m_k > n_k \geq N_1 \),
\[
\frac{a_{m_k} + d_{n_k}}{2} - d_{n_k} \leq \left( 1 - \delta \left( \frac{\epsilon_0}{\text{dist}(C, D) + \epsilon} \right) \right) \text{dist}(C, D) + \epsilon.
\]

Since
\[
\epsilon < \frac{\epsilon_0}{\gamma} - \text{dist}(C, D) \Rightarrow \frac{\epsilon_0}{\text{dist}(C, D) + \epsilon} > \gamma,
\]
which implies \(1 - \delta \left( \frac{\epsilon_n}{\text{dist}(C,D)} + \epsilon \right) < 1 - \delta(\gamma)\),

\[
III \| \frac{a_{n_k} + d_{n_k}}{2} - d_{n_k} \| \leq (1 - \delta(\gamma))(\text{dist}(C,D)) + \epsilon \\
\leq (1 - \delta(\gamma)) \left( \text{dist}(C,D) + \frac{\text{dist}(C,D)}{\delta}(\gamma)1 - \delta(\gamma) \right) \\
< \text{dist}(C,D),
\]

which is a contradiction. Hence, the lemma holds. \(\square\)

Next, we recall few \(p\)-cyclic maps with some contraction conditions imposed on them, which are defined in [3,9,10].

**Definition 2** ([10], Definition 3.1). For a non-empty set \(M\), suppose \(\rho : M \times M \to [0, \infty)\) forms a metric and \(B_1, B_2, ..., B_p\) \((p \geq 2)\) are non-empty subsets of \(M\). Let \(T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i\) be a \(p\)-cyclic map, \(T\) is said to be \(p\)-cyclic contraction, if there exists \(k \in (0,1)\) such that for all \(x \in B_i\) and \(y \in B_{i+1}\), we have

\[
\rho(Tx, Ty) \leq kp(x, y) + (1 - k)\text{dist}(B_i, B_{i+1}), \quad \forall \ i \in \{1,2,\ldots,p\}.
\]

**Definition 3** ([9], Definition 3.5). Let \(B_1, B_2, ..., B_p\) \((p \geq 2)\) be non-empty subsets of a metric space \((M, \rho)\). A \(p\)-cyclic map \(T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i\) is said to be \(p\)-cyclic MK-contraction, if for all \(\epsilon > 0\) there exists \(\delta > 0\) such that whenever \(p(x, y) < \text{dist}(B_i, B_{i+1}) + \epsilon + \delta\), we have

\[
\rho(Tx, Ty) < \text{dist}(B_i, B_{i+1}) + \epsilon,
\]

where \(x \in B_i\), \(y \in B_{i+1}\) and \(1 \leq i \leq p\).

**Definition 4** ([17], Definition 2). Let \(\psi : [0, \infty) \to [0, \infty)\) be a map such that \(\psi(0) = 0\) and \(\psi(\theta) > 0\) if \(\theta > 0\). We say that \(\psi\) is an L-function if for all \(\theta > 0\) there exists \(\delta > 0\) such that \(\psi(t) \leq \theta\) for all \(t \in [\theta, \theta + \delta]\).

**Definition 5** ([3], Definition 2.1). For a non-empty set \(M\), suppose \(\rho : M \times M \to [0, \infty)\) forms a metric and \(C\) and \(D\) are non-empty subsets of \(M\). A cyclic map \(T : C \cup D \to C \cup D\) is said to be cyclic \(\varphi\)-contraction if

\[
\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\text{dist}(C, D)) \quad \text{for all } \ x \in C, \ y \in D,
\]

where \(\varphi : [0, \infty) \to [0, \infty)\) is a strictly increasing map.

4. \(p\)-Cyclic Sequence and \(p\)-Cyclic Complete Metric Space

In this article, \(\mathbb{N}_0\) refers to \(\mathbb{N} \cup \{0\}\). The notion of \(p\)-cyclic sequence is given as follows:

**Definition 6.** For a non-empty set \(M\), suppose \(\rho : M \times M \to [0, \infty)\) forms a metric and \(B_1, B_2, ..., B_p\) \((p \geq 2)\) are non-empty subsets of \(M\).

1. A sequence \(\{x_n\}_{n=1}^{\infty} \subseteq \bigcup_{i=1}^{p} B_i\) is called a \(p\)-cyclic sequence if \(x_{pn+i} \in B_i\), for all \(n \in \mathbb{N}_0\) and \(i = 1, 2, ..., p\).
2. We say that \(\{x_n\}_{n=1}^{\infty}\) is a \(p\)-cyclic Cauchy sequence, if for given \(\epsilon > 0\) there exists an \(N_0 \in \mathbb{N}\) such that for some \(i \in \{1, 2, \ldots, p\}\), we have

\[
\rho(x_{pn+i}, x_{pm+i+1}) < \text{dist}(B_i, B_{i+1}) + \epsilon, \quad \forall \ m, n \geq N_0.
\]
3. A p-cyclic sequence \( \{x_n\}_{n=1}^{\infty} \) in \( \bigcup_{i=1}^{p} B_i \) is said to be p-cyclic bounded, if \( \{x_{pn+i}\}_{n=0}^{\infty} \) is bounded in \( B_i \) for some \( i \in \{1, 2, \ldots, p\} \).

4. Let \( \{x_n\}_{n=1}^{\infty} \) be a p-cyclic sequence in \( \bigcup_{i=1}^{p} B_i \). If for some \( j \in \{1, 2, \ldots, p\} \) the subsequence \( \{x_{pn+j}\} \) of \( \{x_n\}_{n=1}^{\infty} \) converges in \( B_j \), then we say that \( \{x_n\}_{n=1}^{\infty} \) is p-cyclic convergent.

5. Under the assumption that \( B_1, B_2, \ldots, B_p \) are non-empty closed subsets of a metric space \((M, \rho)\), we say that \( \bigcup_{i=1}^{p} B_i \) is p-cyclic complete if every p-cyclic Cauchy sequence in \( \bigcup_{i=1}^{p} B_i \) is p-cyclic convergent.

6. If there are subsets \( B_1, B_2, \ldots, B_p \) (\( p \geq 2 \)) of \((M, \rho)\) such that \( M = \bigcup_{i=1}^{p} B_i \) and \( \bigcup_{i=1}^{p} B_i \) is p-cyclic complete, then we call \((M, \rho)\) is p-cyclic complete.

Remark 1. Note that a p-cyclic sequence that is a Cauchy sequence in the usual sense is a p-cyclic Cauchy sequence. On the other hand, p-cyclic Cauchy sequences need not be Cauchy sequences in the usual sense, even if \( \operatorname{dist}(B_i, B_{i+1}) = 0 \ \forall i \in \{1, 2, \ldots, p\} \).

The following examples illustrate the notion of p-cyclic sequence and p-cyclic Cauchy sequence.

Example 1. Consider \( \mathbb{R} \) with the usual metric. Let \( I_1 = [0, 1] \), \( I_2 = [1, 2] \) and \( I_3 = [2, 3] \). The sequence \( \{x_n\}_{n=1}^{\infty} \) defined by \( x_n = n - 1 \mod 3 + \frac{1}{n} \) for \( n = 1, 2, 3, \ldots \) is a three-cyclic sequence in \( \bigcup_{i=1}^{3} I_i \) but not a three-cyclic Cauchy sequence.

Example 2. Let \( X = \mathbb{R}^2 \) be a Euclidean space. Let the subsets \( B_i \), \( i = 1, 2, 3, 4 \) be as follows:

\[
B_1 = \{(0, 1 + x) : 0 \leq x \leq 1\}, \quad B_2 = \{(1 + x, 0) : 0 \leq x \leq 1\},
\]

\[
B_3 = \{(0, -(1 + x)) : 0 \leq x \leq 1\} \quad \text{and} \quad B_4 = \{(-1 + x, 0) : 0 \leq x \leq 1\}.
\]

Then, \( \operatorname{dist}(B_i, B_{i+1}) = \sqrt{2} \) for \( i = 1, 2, 3, 4 \), where \( B_5 = B_1 \).

Let us define a sequence \( \{(x_n, y_n)\}_{n=1}^{\infty} \) in \( \bigcup_{i=1}^{4} B_i \) as follows:

\[
(x_n, y_n) = \begin{cases} 
(0, 1 + \frac{1}{n^t}), & n = 4k + 1, k = 0, 1, 2, 3, \ldots; \\
(1 + \frac{1}{n^t}, 0), & n = 4k + 2, k = 0, 1, 2, 3, \ldots; \\
(0, -(1 + \frac{1}{n^t})), & n = 4k + 3, k = 0, 1, 2, 3, \ldots; \\
(-1 + \frac{1}{n^t}, 0), & n = 4k + 4, k = 0, 1, 2, 3, \ldots;
\end{cases}
\]

where \( t \in \mathbb{N} \) and \( t > 1 \).

Then, \( \{(x_n, y_n)\}_{n=1}^{\infty} \) is a four-cyclic Cauchy sequence in \( \bigcup_{i=1}^{4} B_i \).

The following Proposition shows that a p-cyclic Cauchy sequence is p-cyclic bounded.

Proposition 2. For a non-empty set \( M \), suppose \( \rho : M \times M \to [0, \infty) \) forms a metric and \( B_1, B_2, \ldots, B_p \) (\( p \geq 2 \)) are non-empty subsets of \( M \). Then, every p-cyclic Cauchy sequence in \( \bigcup_{i=1}^{p} B_i \) is p-cyclic bounded.

Proof. Let \( \{x_n\}_{n=1}^{\infty} \) be a p-cyclic Cauchy sequence in \( \bigcup_{i=1}^{p} B_i \). Then, for some \( i \in \{1, 2, \ldots, p\} \), there exists an \( N \in \mathbb{N} \) such that

\[
\rho(x_{pn+i}, x_{pn+i+1}) < 1 + \operatorname{dist}(B_i, B_{i+1}) \quad \text{for all} \quad n \geq N.
\]

Therefore, for all \( n \in \mathbb{N} \), \( x_{pn+i} \in B(x_{pn+i+1}, r) \) where

\[
r = \max\{1 + \operatorname{dist}(B_i, B_{i+1}), \rho(x_{pn+i}, x_{pn+i+1}) : n = 1, 2, 3, \ldots, N\}.
\]

Thus, \( \{x_{pn+i}\}_{n=0}^{\infty} \) is bounded for some \( i \in \{1, 2, \ldots, p\} \). Hence, \( \{x_n\}_{n=1}^{\infty} \) is p-cyclic bounded. \( \square \)
Assume that \( A \parallel (\text{ii}) \).

(i) For given \( X = \bigcup_{i=1}^{p} B_i \),

\[
B_i = \{ e_{pn+i}, n \in \mathbb{N}_0 \}, i = 1, 2, ..., p,
\]

and \( \{ e_n \} \) is a sequence whose \( n \)th term is 1 and all the other terms are zero. Then, \( B_1, B_2, ..., B_p \) are closed subsets of \( l^{\infty} \) and hence \( (M, \rho) \) is complete. Further, \( \text{dist}(B_i, B_{i+1}) = 1 \) for all \( i = 1, 2, ..., p \). Since

\[
\| e_{pn+i} - e_{pm+i+1} \| = 1 = \text{dist}(B_i, B_{i+1}), \quad n, m \in \mathbb{N}_0, i = 1, 2, ..., p.
\]

Then, the sequence \( \{ e_n \}_{n=1}^{\infty} \) is a \( p \)-cyclic Cauchy sequence in \( X \). However, none of the subsequence \( \{ e_{pn+i} \}_{n=0}^{\infty} \) of \( \{ e_n \}_{n=1}^{\infty} \) converges in \( B_i \) for all \( i = 1, 2, ..., p \). Hence, \( (M, \rho) \) is not \( p \)-cyclic complete.

Remark 2. A complete metric space need not be \( p \)-cyclic complete. For example, let us consider \( (l^{\infty}, \| \cdot \|_{\text{sup}}) \) and let \( X = \bigcup_{i=1}^{p} B_i \), where

\[
B_i = \{ e_{pn+i}, n \in \mathbb{N}_0 \}, i = 1, 2, 3, ..., p,
\]

and \( \{ e_n \} \) is a sequence whose \( n \)th term is 1 and all the other terms are zero. Then, \( B_1, B_2, ..., B_p \) are closed subsets of \( l^{\infty} \) and hence \( (M, \rho) \) is complete. Further, \( \text{dist}(B_i, B_{i+1}) = 1 \) for all \( i = 1, 2, ..., p \). Since

\[
\| e_{pn+i} - e_{pm+i+1} \| = 1 = \text{dist}(B_i, B_{i+1}), \quad n, m \in \mathbb{N}_0, i = 1, 2, ..., p.
\]

Then, the sequence \( \{ e_n \}_{n=1}^{\infty} \) is a \( p \)-cyclic Cauchy sequence in \( X \). However, none of the subsequence \( \{ e_{pn+i} \}_{n=0}^{\infty} \) of \( \{ e_n \}_{n=1}^{\infty} \) converges in \( B_i \) for all \( i = 1, 2, ..., p \). Hence, \( (M, \rho) \) is not \( p \)-cyclic complete.

The following Proposition is an example of two-cyclic complete metric space.

Proposition 3. Let \( A_1 \) and \( A_2 \) be subsets of a uniformly convex Banach space \( X \), which are non-empty and closed. If either \( A_1 \) or \( A_2 \) is convex, then \( A_1 \cup A_2 \) is two-cyclic complete.

Proof. Let \( \{ x_n \} \) be a two-cyclic Cauchy sequence in \( A_1 \cup A_2 \). Then, \( \{ x_{2n+1} \}_{n=0}^{\infty} \subseteq A_1 \) and \( \{ x_{2n} \}_{n=1}^{\infty} \subseteq A_2 \). Assume that \( A_1 \) is convex. Since \( \{ x_n \} \) is a two-cyclic Cauchy sequence in \( A_1 \cup A_2 \), for \( \epsilon > 0 \), there exists an \( n_0 \in \mathbb{N} \) such that

\[
\| x_{2n+1} - x_{2n+2} \| < \text{dist}(A_1, A_2) + \epsilon, \quad m > n \geq n_0.
\]  

(3)

In addition, since \( \| x_{2n+1} - x_{2n+2} \| < \text{dist}(A_1, A_2) + \epsilon, \quad n \geq n_0 \), we have \( \| x_{2n+1} - x_{2n+2} \| \rightarrow \text{dist}(A_1, A_2) \) as \( n \rightarrow \infty \). Let \( n = n_1 = n_0 + 1 \) and \( m = m_1 = n_0 + s + 1 \) where \( s \in \mathbb{N} \) be the first element satisfying Equation (3). Then, \( \{ x_{2n_1+1} \}_{i=1}^{\infty} \) and \( \{ x_{2m_1+1} \}_{i=1}^{\infty} \) are two sequences in \( A_1 \) and \( \{ x_{2n_1+2} \}_{i=1}^{\infty} \) is a sequence in \( A_2 \) satisfying the following:

(i) For given \( \epsilon > 0 \) there exists an \( N_1 \in \mathbb{N} \) such that

\[
\| x_{2n_1+1} - x_{2n_1+2} \| < \text{dist}(A_1, A_2) + \epsilon, \quad m_i > n_i \geq N_1.
\]

(ii) \( \| x_{2n_1+1} - x_{2n_1+2} \| \rightarrow \text{dist}(A_1, A_2) \) as \( i \rightarrow \infty \).

Thus, Conditions (i) and (ii) of Lemma 2 are satisfied. Since \( A_1 \) is convex, by Lemma 2, there exists \( N_2 \in \mathbb{N} \) such that

\[
\| x_{2n_1+1} - x_{2m_1+1} \| \leq \epsilon, \quad m_i > n_i \geq N_2.
\]

Choosing \( N_2 > n_0 + s \), we have

\[
\| x_{2n+1} - x_{2m+1} \| \leq \epsilon, \quad m > n \geq N_2.
\]

Hence, \( \{ x_{2n+1} \}_{n=1}^{\infty} \) is a Cauchy sequence in \( A_1 \). Therefore, \( \{ x_{2n+1} \}_{n=1}^{\infty} \) converges in \( A_1 \). It yields that \( \{ x_n \}_{n=1}^{\infty} \) is two-cyclic convergent and hence \( A_1 \cup A_2 \) is two-cyclic complete.

Similarly, we can prove that \( A_1 \cup A_2 \) is two-cyclic complete if \( A_2 \) is convex. \( \square \)

5. \( p \)-Cyclic Strict Contraction Maps

We introduce a notion of \( p \)-cyclic strict contraction, which is a generalization of strict contraction in the usual sense.
Definition 7. For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_1, B_2, ..., B_p$ ($p \geq 2$) are non-empty subsets of $M$. A $p$-cyclic map $T$ is said to be $p$-cyclic strict contraction if, for all $x \in B_i$, $y \in B_{i+1}$, $1 \leq i \leq p$:

(i) $\rho(x, y) > \text{dist}(B_i, B_{i+1}) \Rightarrow \rho(Tx, Ty) < \rho(x, y)$; and

(ii) $\rho(x, y) = \text{dist}(B_i, B_{i+1}) \Rightarrow \rho(Tx, Ty) = \rho(x, y)$.

Remark 3. Note that, if $B_i = A$, for all $i = 1, 2, ..., p$, then $p$-cyclic strict contraction is a strict contraction in the usual sense. It is clear that the $p$-cyclic strict contraction also forms a $p$-cyclic non-expansive map.

The following Proposition proves an important property of $p$-cyclic strict contraction map.

Proposition 4. For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_1, B_2, ..., B_p$ ($p \geq 2$) are non-empty subsets of $M$. Let $x \in B_i$ ($1 \leq i \leq p$). Suppose that $T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i$ is a $p$-cyclic strict contraction map and if for all $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{pn}x, T^{pn+1}x) < \text{dist}(B_i, B_{i+1}) + \epsilon, \; n, m \geq n_0,$$

then for a given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$d(T^{pn+k}x, T^{pn+k+1}x) < \text{dist}(B_i, B_{i+1}) + \epsilon, \; n, m \geq n_1, \; k \in \{1, 2, \ldots, p\}.$$

Proof. Let $x \in B_i$ be such that $\{T^{pn}x\}$ satisfies Equation (4) ($1 \leq i \leq p$) and let $\epsilon > 0$ be given. Since $T$ is $p$-cyclic non-expansive, for any $m, n \in \mathbb{N}$ with $m > n$, $k \in \{1, 2, \ldots, p\}$, we have

$$d(T^{pn+k}x, T^{pn+k+1}x) \leq d(T^{pn}x, T^{pn+1}x)$$

$$< \text{dist}(B_i, B_{i+1}) + \epsilon, \; n, m \geq n_0 \text{ (by Equation (4))}$$

$$= \text{dist}(A_i, A_{i+1}) + \epsilon, \; n, m \geq n_0 \text{ (by Equation (1))}.$$ 

\[\square\]

6. $\Omega$ Class of Mappings

Many $p$-cyclic maps with various contractive conditions posses some common properties listed in the following definition. Thus, we introduce a notion of class $\Omega$.

Definition 8. For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_1, B_2, ..., B_p$ ($p \geq 2$) are non-empty subsets of $M$. A $p$-cyclic map $T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i$ is said to belong to the class $\Omega$ if

1. $T$ is $p$-cyclic strict contraction.
2. If $x, y \in B_i$, then $\lim_{n \to \infty} \rho(T^{pn}x, T^{pn+1}y) = \text{dist}(B_i, B_{i+1}), 1 \leq i \leq p$.

We list some $p$-cyclic maps that belong to the class $\Omega$. First, we prove that a $p$-cyclic contraction map, which is defined in [10] (Definition 2), belongs to the class $\Omega$.

Example 3. Let $B_1, B_2, ..., B_p$ be non-empty subsets of a metric space $(M, \rho)$. Let $T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i$ is a $p$-cyclic contraction map. Then, $T \in \Omega$. 
Proof. Because the the map $T$ is a $p$-cyclic contraction, we have
\[
\rho(Tx, Ty) \leq kp(x, y) + (1-k)\text{dist}(B_i, B_{i+1}), \quad x \in B_i, y \in B_{i+1}, \ i = 1, 2, ..., p,
\]
for some $k \in (0, 1)$. If $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then $\rho(Tx, Ty) = \rho(x, y)$. In addition, if $\rho(x, y) > \text{dist}(B_i, B_{i+1})$, then
\[
\rho(Tx, Ty) < kp(x, y) + (1-k)\rho(x, y) = \rho(x, y).
\]

Therefore, $T$ is $p$-cyclic strict contraction. The second condition of Definition 8 follows from Lemma 3.3 in [10]. Hence, $T \in \Omega$. □

Next, we prove that the $p$-cyclic Meir–Keeler map ($p$-cyclic MK-map) introduced in [9] (Definition 3) belongs to the class $\Omega$.

Example 4. For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_1, B_2, ..., B_p$ ($p \geq 2$) are non-empty subsets of $M$. Let $T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i$ be a $p$-cyclic MK-contraction map. Then, $T \in \Omega$.

Proof. From Remark 3.6 in [9], we know that $T$ is a $p$-cyclic MK-contraction if and only if there is an $L$-function $\psi$ such that for all $x \in B_i, y \in B_{i+1}$ and $1 \leq i \leq p$, the following conditions hold:

(i) If $\rho(x, y) > \text{dist}(B_i, B_{i+1}) > 0$, then $\rho(Tx, Ty) - \text{dist}(B_i, B_{i+1}) < \psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) \leq \rho(x, y) - \text{dist}(B_i, B_{i+1})$.

(ii) If $\rho(x, y) - \text{dist}(B_i, B_{i+1}) = 0$, then $\rho(Tx, Ty) - \text{dist}(B_i, B_{i+1}) = 0$.

Therefore, $T$ is $p$-cyclic strict contraction and Condition (2) of Definition 8 follows from Lemma 3.8 in [9]. Thus, $T \in \Omega$. □

Now, we prove that cyclic $\varphi$-contraction introduced in [3] (Definition 5) belongs to the class $\Omega$.

Example 5. For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $C, D$ are non-empty subsets of $M$. Let $T : C \cup D \to C \cup D$ be a cyclic $\varphi$-contraction map. Then, $T \in \Omega$.

Proof. Let $x \in C$ and $y \in D$. If $\rho(x, y) > \text{dist}(C, D)$, since $\varphi$ is strictly increasing,
\[
\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\text{dist}(C, D)) < \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) = \rho(x, y).
\]

If $\rho(x, y) = \text{dist}(C, D)$, then $\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) = \rho(x, y)$. Therefore, we get $\rho(x, y) = \text{dist}(C, D) \leq \rho(Tx, Ty) \leq \rho(x, y)$. It yields that $\rho(Tx, Ty) = \rho(x, y)$. Thus, $T$ is a two-cyclic strict contraction and Condition (2) of Definition 8 follows from Theorem 3 in [3] (by putting $d_n = \rho(T^{2n}x, T^{2n+1}y), \ x, y \in C$ or $x, y \in D$). Hence, $T \in \Omega$. □

Next, we establish an example of $p$-cyclic map satisfying a contraction condition of Geraghty’s type [18] and show that it belongs to the class $\Omega$. Here, we use a class of functions $S$ introduced by
Geraghty [18], where, if $S$ is the class of all functions $\vartheta : [0,\infty) \to [0,1)$ that satisfies $\vartheta(t_n) \to 1$, then $t_n \to 0$, $t_n \in [0,\infty)$ for $n \in \mathbb{N}$.

**Example 6.** For a non-empty set $M$, suppose $\rho : M \times M \to [0,\infty)$ forms a metric and $B_1, B_2, \ldots, B_p$ ($p \geq 2$) are non-empty subsets of $M$. Let $T : \bigcup_{i=1}^p B_i \to \bigcup_{i=1}^p B_i$ be a $p$-cyclic map such that for some $\vartheta \in S$,

$$
\rho(Tx,Ty) \leq \vartheta(\rho(x,y))\rho(x,y) + (1 - \vartheta(\rho(x,y)))\text{dist}(B_i,B_{i+1}), \ x \in B_i, y \in B_{i+1}.
$$

Then,

(a) $T$ is a $p$-cyclic strict contraction.

(b) $\lim_{n \to \infty} \rho(T^{pn}x,T^{pn+1}y) = \text{dist}(B_i,B_{i+1}), \ x,y \in B_i$.

**Proof.** (a) Let $x \in B_i, y \in B_{i+1}$.

Case (1): If $\rho(x,y) > \text{dist}(B_i,B_{i+1})$, we have

$$
\rho(Tx,Ty) \leq \vartheta(\rho(x,y))\rho(x,y) + (1 - \vartheta(\rho(x,y)))\text{dist}(B_i,B_{i+1})
= \vartheta(\rho(x,y))\rho(x,y) + \text{dist}(B_i,B_{i+1}) - \vartheta(\rho(x,y))\text{dist}(B_i,B_{i+1})
= \vartheta(\rho(x,y))[\rho(x,y) - \text{dist}(B_i,B_{i+1})] + \text{dist}(B_i,B_{i+1}), (*)
< \rho(x,y) - \text{dist}(B_i,B_{i+1}) + \text{dist}(B_i,B_{i+1}) \quad \text{(because $\vartheta(\rho(x,y)) < 1$)}
= \rho(x,y).
$$

Case (2): If $\rho(x,y) = \text{dist}(B_i,B_{i+1})$, then from (*),

$$
\rho(Tx,Ty) \leq \rho(x,y).
$$

By Equation (1), $\rho(x,y) = \text{dist}(B_i,B_{i+1}) = \text{dist}(B_{i+1},B_{i+2}) \leq \rho(Tx,Ty) \leq \rho(x,y)$, therefore

$$
\rho(Tx,Ty) = \rho(x,y).
$$

Hence, $T$ is $p$-cyclic strict contraction.

(b) Let $x,y \in B_i$. Since $T$ is $p$-cyclic non-expansive, $\{\rho(T^{pn}x,T^{pn+1}y)\}$ is a decreasing sequence and is bounded below by $\text{dist}(B_i,B_{i+1})$. Therefore,

$$
\rho(T^{pn}x,T^{pn+1}y) \to r \text{ as } n \to \infty \quad \text{and} \quad r \geq \text{dist}(B_i,B_{i+1}),
$$

where $r = \inf_{n \geq 1} \rho(T^{pn}x,T^{pn+1}y)$.

Claim: $r = \text{dist}(B_i,B_{i+1})$.

If $\rho(T^{pn}x,T^{pn+1}y) = \text{dist}(B_i,B_{i+1})$ for some $n$, then by the $p$-cyclic non-expansiveness of $T$,

$$
\rho(T^{pn+k}x,T^{pn+k+1}y) = \rho(T^{pn}x,T^{pn+1}y), \ k = 1,2,\ldots.
$$

Hence, we have

$$
\rho(T^{pn}x,T^{pn+1}y) \to \text{dist}(B_i,B_{i+1}) \text{ as } n \to \infty.
$$

Let us assume that $\rho(T^{pn}x,T^{pn+1}y) > \text{dist}(B_i,B_{i+1}), \ n \in \mathbb{N}$.
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Proof. (a) Let $\rho$ be a p-cyclic non expansive map, then
\[
\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \rho(T^{pn+1}x, T^{pn+2}y) \\
\leq \theta(\rho(T^{pn}x, T^{pn+1}y) + (1 - \theta)\rho(T^{pn+1}y, T^{pn+2}y))\text{dist}(B_i, B_{i+1}),
\]
\[
\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1}) \leq \theta(\rho(T^{pn}x, T^{pn+1}y)) \\
\quad \text{[}\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})\text{]}
\]
Since $\theta \in \mathbb{S}$,
\[
\frac{\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1})}{\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})} \leq \theta(\rho(T^{pn}x, T^{pn+1}y)) < 1. \tag{5}
\]

Since $r = \lim_{n \to \infty} \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) > \text{dist}(B_i, B_{i+1})$ by our assumption, letting $n \to \infty$ in Equation (5), we get
\[
1 \leq \lim_{n \to \infty} \theta(\rho(T^{pn}x, T^{pn+1}y)) \leq 1,
\]
that is, $\lim_{n \to \infty} \theta(\rho(T^{pn}x, T^{pn+1}y)) = 1$. However, $\lim_{n \to \infty} \rho(T^{pn}x, T^{pn+1}y) = r > 0$, which contradicts $\theta \in \mathbb{S}$. Hence, $r = \text{dist}(B_i, B_{i+1})$. This proves Part (b). \qed

Finally, an example of a p-cyclic map that is of Boyd–Wong type \cite{19} and belongs to the class $\Omega$ is given:

Example 7. Let $M$ be a non-empty set equipped with a metric $\rho$. Suppose that $B_1, B_2, ..., B_p$ ($p \geq 2$) are non-empty subsets of $M$. Suppose that $\psi : [0, \infty) \to [0, \infty]$ is upper semi-continuous from the right and satisfies $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. Suppose also that $T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i$ is a p-cyclic map. Suppose
\[
\rho(Tx, Ty) \leq \psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) + \text{dist}(B_i, B_{i+1}), \tag{6}
\]
x $\in B_i$, $y \in B_{i+1}$, $1 \leq i \leq p$.
Then, the following conditions hold:
\begin{enumerate}[(a)]
  \item $T$ is p-cyclic strict contraction.
  \item For $x, y \in B_i$, $\rho(T^{pn}x, T^{pn+1}y) \to \text{dist}(B_i, B_{i+1})$, as $n \to \infty$.
\end{enumerate}

Proof. (a) Let $x \in B_i$ and $y \in B_{i+1}$.
Case (i): If $\rho(x, y) > \text{dist}(B_i, B_{i+1})$. Since $\psi(t) < t$ for $t > 0$, by Equation (6), we have
\[
\rho(Tx, Ty) < \rho(x, y) - \text{dist}(B_i, B_{i+1}) + \text{dist}(B_i, B_{i+1}) \\
= \rho(x, y).
\]
Case (ii): If $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then $\psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) = 0$. Therefore, by Equation (6), $\rho(Tx, Ty) = \text{dist}(B_i, B_{i+1})$. That is, $\rho(Tx, Ty) = \rho(x, y)$.
(b) Let $x, y \in B_i$. Note that
\[
\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \rho(T^{pn}x, T^{pn+1}y), \ n \in \mathbb{N}.
\]
Then, the sequence $\{\rho(T^{pn}x, T^{pn+1}y)\}_{n=1}^{\infty}$ is bounded below by $\text{dist}(B_i, B_{i+1})$ and non-increasing sequence. Hence, $\rho(T^{pn}x, T^{pn+1}y) \to r$ as $n \to \infty$ and $r \geq \text{dist}(B_i, B_{i+1})$, where $r = \inf_{n \geq 1} \{\rho(T^{pn}x, T^{pn+1}y)\}$.
Theorem 1. For a non-empty set M, suppose 

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non-empty subsets of M.

Claim: \( r = \text{dist}(B_i, B_{i+1}) \).

Case (1): If \( \rho(T^n x, T^{n+1} y) = \text{dist}(B_i, B_{i+1}) \) for some \( n \in \mathbb{N} \).

Then, by the \( p \)-cyclic non-expansiveness of \( T \),

\[
\rho(T^{pn+k} x, T^{pn+k+1} y) = \text{dist}(B_i, B_{i+1}), \quad k = 1, 2, \ldots .
\]

Thus, \( \rho(T^n x, T^{n+1} y) \to \text{dist}(B_i, B_{i+1}) \), as \( n \to \infty \).

Case (2): If \( \rho(T^n x, T^{n+1} y) > \text{dist}(B_i, B_{i+1}) \) for all \( n \in \mathbb{N} \).

Since \( T \) is \( p \)-cyclic non-expansive,

\[
\rho(T^{(n+1)} x, T^{(n+1)+1} y) \leq \rho(T^{pn+1} x, T^{pn+2} y) \leq \psi(\rho(T^{pn} x, T^{pn+1} y) - \text{dist}(B_i, B_{i+1})) + \text{dist}(B_i, B_{i+1}),
\]

\[
\rho(T^{(n+1)} x, T^{(n+1)+1} y) - \text{dist}(B_i, B_{i+1}) \leq \psi(\rho(T^{pn} x, T^{pn+1} y) - \text{dist}(B_i, B_{i+1})).
\]

Taking the \( \lim \sup \) on both sides,

\[
\lim_{n \to \infty} \rho(T^{(n+1)} x, T^{(n+1)+1} y) - \text{dist}(B_i, B_{i+1}) \leq \limsup_{n \to \infty} \psi(\rho(T^{pn} x, T^{pn+1} y) - \text{dist}(B_i, B_{i+1})).
\]

Since \( \rho(T^{(n+1)} x, T^{(n+1)+1} y) - \text{dist}(B_i, B_{i+1}) \downarrow r - \text{dist}(B_i, B_{i+1}) \) and \( \psi \) is upper semi-continuous from the right,

\[
r - \text{dist}(B_i, B_{i+1}) \leq \psi(r - \text{dist}(B_i, B_{i+1})).
\]

Let \( t = r - \text{dist}(B_i, B_{i+1}) \). If \( r > \text{dist}(B_i, B_{i+1}) \), then \( t > 0 \) and also \( t \leq \psi(t) \), which is a contradiction to the definition of \( \psi \). Hence, \( r = \text{dist}(B_i, B_{i+1}) \). \( \square \)

In a similar way, one may check whether some other known/unknown \( p \)-cyclic maps belong to the class \( \Omega \).

7. Best Proximity Point Results of \( \Omega \) Class of Mappings

Below, we give a convergence result for \( \Omega \) class of mappings.

**Theorem 1.** For a non-empty set \( M \), suppose \( \rho : M \times M \to [0, \infty) \) forms a metric and \( B_1, B_2, \ldots, B_p \) (\( p \geq 2 \)) are non-empty subsets of \( M \).

Let \( T : \bigcup_{i=1}^{p} B_i \to \bigcup_{i=1}^{p} B_i \) be a \( p \)-cyclic map that belongs to the class \( \Omega \). Assume for some \( k \in \mathbb{N} \) and \( x \in B_i \) (\( 1 \leq i, k \leq p \)), \( \{T^{pn+k} x\} \) converges to \( \xi \in B_{i+k} \). Then, \( \xi \) is a best proximity point of \( T \) in \( B_{i+k} \).

**Proof.** Let \( x \in B_i \) be as given in the theorem. By Equation (1), for each \( n \in \mathbb{N} \), we have,

\[
\text{dist}(B_{i+k}, B_{i+k+1}) = \text{dist}(B_{i+k-1}, B_{i+k}) \leq \rho(T^{pn+k-1} x, \xi) \leq \rho(T^{pn+k-1} x, T^{pn+k} x) + \rho(T^{pn+k} x, \xi).
\]

Since \( T \in \Omega \),

\[
\lim_{n \to \infty} (\rho(T^{pn+k-1} x, T^{pn+k} x) + \rho(T^{pn+k} x, \xi)) = \text{dist}(B_{i+k-1}, B_{i+k}).
\]
Theorem 2. For a non-empty set M, suppose $\rho(T^{p}x, T^{p+1}x) = \text{dist}(B_{i+k-1}, B_{i+k}) = \text{dist}(B_{i+k}, B_{i+k+1})$.  

Now, 

$$\text{dist}(B_{i+k}, B_{i+k+1}) \leq \rho(\xi, T\xi) = \lim_{n \to \infty} \rho(T^{pn+k}x, T^{pn+1}x) \leq \lim_{n \to \infty} \rho(T^{pn+k-1}x, \xi) = \text{dist}(B_{i+k}, B_{i+k+1})$$

Hence, $\rho(\xi, T\xi) = \text{dist}(B_{i+k}, B_{i+k+1})$.  \hfill \Box$

Now, we prove the existence of best proximity point for mappings which belong to class $\Omega$ defined on a $p$-cyclic complete metric space.

**Theorem 2.** For a non-empty set $M$, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_{1}, B_{2}, \ldots, B_{p}$ ($p \geq 2$) are non-empty subsets of $M$. Suppose that $M = \bigcup_{i=1}^{p} B_{i}$ and $\bigcup_{i=1}^{p} B_{i}$ is $p$-cyclic complete. Let $T : \bigcup_{i=1}^{p} B_{i} \to \bigcup_{i=1}^{p} B_{i}$ be a $p$-cyclic map which belongs to the class $\Omega$. Then, there exists a best proximity point of $T$ in $B_{j}$ for some $j \in \{1, 2, \ldots, p\}$.

**Proof.** Let $x \in B_{i}, 1 \leq i \leq p$.

Define a sequence $\{x_{n}\}_{n=1}^{\infty}$ in $(M, \rho)$ by

$$x_{n} := T^{n}x \text{ for } n \in \mathbb{N}.$$ 

Claim: $\{T^{n}x\}_{n=1}^{\infty}$ is a $p$-cyclic Cauchy sequence.

Let $m, n \in \mathbb{N}$ be such that $m > n$,

$$\rho(T^{pn+r}x, T^{pn+1}y) = \rho(T^{pn+r}y, T^{pn+1}x), \text{ where } m = n + r, \ r \in \mathbb{N}$$

$$\rho(T^{pn+r}x, T^{pn+1}x) = \rho(T^{pn+r}y, T^{pn+1}y), \text{ where } y = T^{pr}x \in B_{i}$$

$\Rightarrow \text{dist}(B_{j}, B_{i+1})$, as $n \to \infty$ (because $T \in \Omega$).

This implies that, for all $\epsilon > 0$, there exists an $n_{0} \in \mathbb{N}$ such that

$$\rho(T^{pn+r}x, T^{pn+1}y) < \epsilon + \text{dist}(\{B_{j}, B_{i+1}\}), \text{ } m, n \geq n_{0}.$$ 

By Proposition 4, for a given $\epsilon > 0$, there exists an $n_{1} \in \mathbb{N}$ such that

$$d(T^{pn+k}x, T^{pn+k+1}x) < \epsilon + d(A_{i+k}, A_{i+k+1}), \text{ } m, n \geq n_{1}, \ k \in \{1, 2, \ldots, p\}.$$ 

Therefore, the sequence $\{T^{n}x\}$ is a $p$-cyclic Cauchy sequence in $(M, \rho)$. Since $(M, \rho)$ is $p$-cyclic complete, there exists a $k \in \{1, 2, \ldots, p\}$ such that $\{T^{pn+k}x\}$ converges to $z \in A_{i+k}$. By Theorem 1, $z$ is best proximity point of $T$ in $B_{j}$, where $j = i + k$.  \hfill \Box

**Remark 4.** In Theorem 2, suppose that $\text{dist}(B_{i}, B_{i+1}) = 0$, for some $i \in \{1, 2, \ldots, p\}$. Then, by Equation (1), $\text{dist}(B_{k}, B_{k+1}) = 0$, $1 \leq k \leq p$. This implies that $Tz = z$, that is, $z$ is a fixed point of $T$. Since $T$ is $p$-cyclic, $z \in \bigcap_{i=1}^{p} B_{i}$ and hence $\bigcap_{i=1}^{p} B_{i}$ is non-empty.
To prove the uniqueness of fixed point of $T$, let $\alpha, \beta \in B_j$ be such that $\alpha = T\alpha$, $\beta = T\beta$ and $x\alpha \neq \beta$. Then, $p(\alpha, \beta) > 0 = \text{dist}(B_i, B_{i+1})$. Since $T$ is $p$-cyclic strict contraction,

$$p(\alpha, \beta) = p(T\alpha, T\beta) < p(\alpha, \beta),$$

which is a contradiction. Hence, $\alpha = \beta$. This shows that there exists a unique fixed point for $\Omega$ class of mappings in a $p$-cyclic complete metric space.

**Theorem 3.** Let $X$ be a strictly convex normed linear space. Let $B_1, B_2, \ldots, B_p \ (p \geq 2)$ be non-empty, closed, convex subsets of $X$ such that $X = \bigcup_{i=1}^{p} B_i$ and $\bigcup_{i=1}^{p} B_i$ is $p$-cyclic complete. Let $T : \bigcup_{i=1}^{p} B_i \rightarrow \bigcup_{i=1}^{p} B_i$ be a $p$-cyclic map which belongs to the class $\Omega$. Then, for each $j \in \{1, 2, \ldots, p\}$, there exists a unique best proximity point $z_j \in A_j$. In addition, $z_j$ is a unique periodic point of $T$ in $B_j$ and $T^k z_j$ is the unique best proximity point of $T$ in $B_{j+k}$, $k = 0, 1, 2, \ldots, (p-1)$.

**Proof.** From the proof of Theorem 2, for any $x \in B_i, i \in \{1, 2, \ldots, p\}$, $\{T^{pn+q}x\}$ converges to $z \in B_j$, $j = i + q$, for some $q \in \{1, 2, \ldots, p\}$ and by Theorem 1, $z$ is a best proximity point of $T$ in $B_j$. Since $X$ is a strictly convex, by Proposition 1, $z$ is unique.

By Lemma 1, since $T$ is $p$-cyclic non expansive map, $T^k z$ is the best proximity point of $T$ in $B_{j+k}$ for $k = 1, 2, \ldots, (p-1)$.

To prove $z$ is a periodic point of $T$ in $B_j$, it is enough to prove $\|T^p z - Tz\| = \text{dist}(B_j, B_{j+1})$. Suppose $\|T^p z - Tz\| > \text{dist}(B_j, B_{j+1})$. Since $T$ is $p$-cyclic strict contraction,

$$\|T^{p+1} z - T^2 z\| < \|T^p z - Tz\| \leq \|T^{p-1} z - z\| = \lim_{n} \|T^{p-1} z - T^{pn+q} x\| \leq \lim_{n} \|z - T^{(pn+q)-(p-1)} x\| = \lim_{n} \|z - T^{(n-1)+q+1} x\| = \lim_{n} \|T^{p(n-1)+q} x - T^{p(n-1)+q+1} x\| = \text{dist}(B_i, B_{i+q+1}) \ (\text{because } T \in \Omega) = \text{dist}(B_j, B_{j+1}) \ (\text{by Equation (1)}) = \text{dist}(B_{j+1}, B_{j+2})$$

which is a contradiction. Hence, $\|T^p z - Tz\| = \text{dist}(B_j, B_{j+1})$. Since $X$ is strictly convex, $B_j$ is convex and $\|z - Tz\| = \text{dist}(B_j, B_{j+1})$, then $T^p z = z$, that is $z$ is a periodic point of $T$ in $B_j$. To prove the uniqueness of $z$, let $\xi \in B_j$ be such that $T^p \xi = \xi$, then $\{T^{pn} \xi\}$ converges to $\xi \in B_j$. By Theorem 1, $\xi$ is a best proximity point of $T$ in $B_j$. Since $z$ is the unique best proximity point of $T$ in $A_j$, we have $\xi = z$. Hence, the theorem holds. $\square$

8. Conclusions

In this paper, we have introduced a notion of $p$-cyclic Cauchy sequence, which is weaker than the notion of Cauchy sequence in the usual sense. If one subsequence of a $p$-cyclic Cauchy sequence converges, then we say that the $p$-cyclic sequence is $p$-cyclic convergent. If all $p$-cyclic Cauchy sequences converge, then we call the underlying metric space as $p$-cyclic complete metric space. We have shown that a complete metric space need not be $p$-cyclic complete. A class of mappings called $\Omega$ is introduced. The existence of
fixed point and best proximity point of mappings of class $\Omega$ is guaranteed in a $p$-cyclic complete metric space. Many $p$-cyclic maps with various contractive conditions introduced in the literature fall under class $\Omega$, where the best proximity point for such maps were obtained in a uniformly convex Banach space, whereas we have obtained a unique best proximity point in a strictly convex norm linear space. Thus, our main result is a natural generalization of main results of Al-Thagafi [3], Boyd [19], Eldred [5], Geraghty [18], Karpagam [10], Karpagam [9], Kirk [2], Meir [20].

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**References**


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