Neutrosophic Extended Triplet Group Based on Neutrosophic Quadruple Numbers

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Abstract: In this paper, we explore the algebra structure based on neutrosophic quadruple numbers. Moreover, two kinds of degradation algebra systems of neutrosophic quadruple numbers are introduced. In particular, the following results are strictly proved: (1) the set of neutrosophic quadruple numbers with a multiplication operation is a neutrosophic extended triplet group; (2) the neutral element of each neutrosophic quadruple number is unique and there are only sixteen different neutral elements in all of neutrosophic quadruple numbers; (3) the set which has same neutral element is closed with respect to the multiplication operator; (4) the union of the set which has same neutral element is a partition of four-dimensional space.

Keywords: neutrosophic extended triplet group; neutrosophic quadruple numbers; neutrosophic set

1. Introduction

The notion of a neutrosophic set is proposed by F. Smarandache [1] in order to solve real-world problems and some in-depth analysis and research have been carried out [2–5]. Recently, Smarandache and Ali in [6] proposed a new algebraic system, neutrosophic triplet group (NTG), which different from classical groups. From the original definition of NTG, the neutral element is different from the classical algebraic unit element. By removing this restriction, the neutrosophic extended triplet group (NETG) is proposed in [7,8] and the classical group is regarded as a special case of NETG.

As a new algebraic structure, NTG (NETG) immediately attracted the attention of scholars and conducted in-depth research. These studies are mainly carried out by the following three aspects. Firstly, the structure properties of NTG (NETG) have been studied deeply. For examples, paper [8] has conducted an in-depth analysis of the nature of NTG, and the properties and structural features of NTG are studied by using theoretical analysis and software calculations. In paper [9], the notion of the neutrosophic triplet coset and its relation with the classical coset are proposed and the properties of the neutrosophic triplet cosets are given. The neutrosophic duplet sets, neutrosophic duplet semi-groups, and cancellable neutrosophic triplet groups are proposed and the characterizations of cancellable weak neutrosophic duplet semi-groups are established in paper [10]. In order to explore the structure of the algebraic system \((Z_n, \otimes)\), where \(\otimes\) is the classical mod multiplication, paper [11] reveals that for each \(n \in Z^+, n \geq 2, (Z_n, \otimes)\) is a commutative NETG if and only if the factorization of \(n\) is a product of single factors. Moreover, the generalized neutrosophic extended triplet group (GNETG) is proposed in [11] and verify that for each \(n \in Z^+, n \geq 2, (Z_n, \otimes)\) is a commutative GNETG. Secondly, it is the application research on the algebraic system NET. For example, In paper [12], the distinguishing features between an NTG and other algebraic structures are investigated and the first isomorphism theorem was established for NTGs, furthermore, applications of the results on NTG to management
the notion of neutrosophic triplet v-generalized metric space are introduced in [19]. Paper [20], we will reveal that

(proposition, was introduced by Florentin Smarandache in [22]. The algebra system
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where \( T \) extended triplet group (NETG), if the following conditions are satisfied:

\[ \text{Definition 1 ([7,8])}. \] Let \( N \) be a non-empty set together with a binary operation \( * \). Then, \( N \) is called a neutrosophic extended triplet set if for any \( a \in N \), there exists a neutral of “\( a \)” (denote by neut\( (a) \)), and an opposite of “\( a \)” (denote by anti\( (a) \)), such that neut\( (a) \in N \), anti\( (a) \in N \) and:

\[ a \ast \text{neut}(a) = \text{neut}(a) \ast a = a, \quad a \ast \text{anti}(a) = \text{anti}(a) \ast a = \text{neut}(a). \]

The triplet \((a, \text{neut}(a), \text{anti}(a))\) is called a neutrosophic extended triplet.

\[ \text{Definition 2 ([7,8])}. \] Let \((N, \ast)\) be a neutrosophic extended triplet set. Then, \( N \) is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:

1. \((N, \ast)\) is well-defined, i.e., for any \( a, b \in N \), one has \( a \ast b \in N \).
2. \((N, \ast)\) is associative, i.e., \((a \ast b) \ast c = a \ast (b \ast c)\) for all \( a, b, c \in N \).

A NETG \( N \) is called a commutative NETG if for all \( a, b \in N, a \ast b = b \ast a \).

\[ \text{Proposition 1 ([8])}. \] Let \((N, \ast)\) be a NETG. We have:

1. \(\text{neut}(a)\) is unique for any \( a \in N \).
2. \(\text{neut}(a) \ast \text{neut}(a) = \text{neut}(a)\) for any \( a \in N \).
3. \(\text{neut}(\text{neut}(a)) = \text{neut}(a)\) for any \( a \in N \).

\[ \text{Definition 3 ([22,23])}. \] A neutrosophic quadruple number is a number of the form \((a, bT, cI, dF)\), where \( T, I, F \) have their usual neutrosophic logic meanings and \( a, b, c, d \in \mathbb{R} \) or \( \mathbb{C} \). The set \( NQ \), defined by

\[ NQ = \{ (a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C} \}. \] (1)
is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number \((a, b, T, I, F)\), \(a\) is called the known part and \((b, T, I, F)\) is called the unknown part.

**Definition 4** ([22,23]). Let \(N\) be a set, endowed with a total order \(a \prec b\), named “\(a\) prevailed by \(b\)” or “\(a\) less stronger than \(b\)” or “\(a\) less preferred than \(b\)”. We consider \(a \preceq b\) as “\(a\) prevailed by or equal to \(b\)” “\(a\) less stronger than or equal to \(b\)”, or “\(a\) less preferred than or equal to \(b\)”.

For any elements \(a, b \in N\), with \(a \preceq b\), one has the absorbance law:

\[
a \cdot b = b \cdot a = \text{absorb}(a, b) = \max(a, b) = b,
\]

which means that the bigger element absorbs the smaller element. Clearly,

\[
a \cdot a = a^2 = \text{absorb}(a, a) = \max(a, a) = a.
\]

and

\[
a_1 \cdot a_2 \cdot \cdots \cdot a_n = \max(a_1, a_2, \cdots, a_n).
\]

Analogously, we say that “\(a \succ b\)” and we read: “\(a\) prevails to \(b\)” or “\(a\) is stronger than \(b\)” or “\(a\) is preferred to \(b\)”. Also, \(a \succeq b\), and we read: “\(a\) prevails or is equal to \(b\)” “\(a\) is stronger than or equal to \(b\)”, or “\(a\) is preferred or equal to \(b\)”.

**Definition 5** ([22,23]). Consider the set \(\{T, I, F\}\). Suppose in an optimistic way we consider the prevalence order \(T \succ I \succ F\). Then we have: \(T I = IT = \max(T, I) = T\), \(TF = FT = \max(T, F) = T\), \(IF = FI = \max(I, F) = I\), \(TT = T^2 = T\), \(II = I^2 = I\), \(FF = F^2 = F\).

Analogously, suppose in a pessimistic way we consider the prevalence order \(T \prec I \prec F\). Then we have: \(TI = IT = \max(T, I) = I\), \(TF = FT = \max(T, F) = F\), \(IF = FI = \max(I, F) = F\), \(TT = T^2 = T\), \(II = I^2 = I\), \(FF = F^2 = F\).

**Definition 6** ([22,23]). Let \(a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ\). Suppose in an pessimistic way, the neutrosophic expert considers the prevalence order \(T \prec I \prec F\). Then the multiplication operation is defined as following:

\[
a \ast b = (a_1, a_2T, a_3I, a_4F) \ast (b_1, b_2T, b_3I, b_4F)
= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I,
(a_1b_4 + a_2b_4 + a_3b_1 + a_4b_4 + a_4b_1 + a_4b_3 + a_4b_4)F).
\]

Suppose in an optimistic way the neutrosophic expert considers the prevalence order \(T \succ I \succ F\). Then:

\[
a \ast b = (a_1, a_2T, a_3I, a_4F) \ast (b_1, b_2T, b_3I, b_4F)
= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2 + a_3b_1 + a_3b_2 + a_3b_3 + a_4b_4)T,
(a_1b_3 + a_2b_3 + a_3b_1 + a_4b_3 + a_4b_4)I, (a_1b_4 + a_4b_1 + a_4b_4)F).
\]

**Proposition 2** ([22,23]). Let \(NQ = \{(a, b, T, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\}\). We have:
(1) \((NQ, \ast)\) is a commutative monoid.
(2) \((NQ, \ast)\) is a commutative monoid.

3. Main Results

From Proposition 2, we can see that \((NQ, \ast)\) or \((NQ, \ast)\) be a commutative monoid. In these section, we will show that the algebra system \((NQ, \ast)\) or \((NQ, \ast)\) is a NETG.

**Theorem 1.** For the algebra system \((NQ, \ast)\), for every element \(a \in NQ\), there exists the neutral element \(\text{neut}(a)\) and opposite element \(\text{anti}(a)\).
Proof analysis: the proof of this theorem contains two aspects. Firstly, given an element \( a \in NQ, a = (a_1, a_2 T, a_3 I, a_4 F), a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\} \). Being \( a_i \) can select every element in \( \mathbb{R} \), we should discuss from different cases and in each case netu\((a)\) and anti\((a)\) should given. Secondly, we should prove that all the cases discussed above include all the elements in NQ.

**Proof.** Let \( a = (a_1, a_2 T, a_3 I, a_4 F) \), we consider \( a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\} \) and the same results can be gotten when \( a_i \in \mathbb{C} \).

Set netu\((a)\) = \((b_1, b_2 T, b_3 I, b_4 F)\), \( b_i \in \mathbb{R}, i \in \{1, 2, 3, 4\} \) and anti\((a)\) = \((c_1, c_2 T, c_3 I, c_4 F)\), \( c_i \in \mathbb{R}, i \in \{1, 2, 3, 4\} \). From Definition 1 we can get \( a \ast netu(a) = a \), that is \( a_1 b_1 = a_1 \) should hold. So we discuss from two cases, \( a_1 = 0 \) or \( a_1 \neq 0 \).

Case A: when \( a_1 = 0 \).

In this case, we have \( a = (0, a_2 T, a_3 I, a_4 F) \). From Definition 1, \( a \ast anti(a) = netu(a) \), that is \( 0 \cdot c_1 = b_1 \), so we have \( b_1 = 0 \), i.e., \( netu(a) = (0, b_2 T, b_3 I, b_4 F) \). Moreover, from \( a \ast netu(a) = a \), we have \((0, a_2 T, a_3 I, a_4 F) \ast (0, b_2 T, b_3 I, b_4 F) = (0, a_2 T, a_3 I, a_4 F) \), so we have \( a_2 b_2 = a_2 \). So we discuss from \( a_2 = 0 \) or \( a_2 \neq 0 \).

Case A1: \( a_1 = 0, a_2 = 0 \). That is, \( a = (0, 0, a_3 I, a_4 F), netu(a) = (0, b_2 T, b_3 I, b_4 F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a \ast anti(a) = netu(a) \), we have \( 0 c_1 + 0 (c_1 + c_2) = b_2 \), so \( b_2 = 0 \), i.e., \( netu(a) = (0, 0, b_3 I, b_4 F) \). From \((0, 0, a_3 I, a_4 F) \ast (0, 0, b_3 I, b_4 F) = (0, 0, a_3 I, a_4 F) \), we have \( a_3 b_3 = a_3 \). So we discuss from \( a_3 = 0 \) or \( a_3 \neq 0 \).

Case A11: \( a_1 = a_2 = a_3 \neq 0 \), that is, \( a = (0, 0, 0, a_4 F), netu(a) = (0, 0, b_3 I, b_4 F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). In the same way, from \( a \ast anti(a) = netu(a) \), we have \( b_3 = 0 \), i.e., \( netu(a) = (0, 0, 0, b_4 F) \). From \((0, 0, 0, a_4 F) \ast (0, 0, 0, b_4 F) = (0, 0, 0, a_4 F) \), we have \( a_3 b_4 = a_4 \). So we discuss from \( a_4 = 0 \) or \( a_4 \neq 0 \).

Case A111: \( a_1 = a_2 = a_3 = a_4 = 0 \), that is, \( a = (0, 0, 0, 0) \), in this case, we can easily get \( netu(a) = (0, 0, 0, 0) \) and \( anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \), \( c_1 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A12: \( a_1 = a_2 = a_3 \neq 0, a_4 = 0 \), being that \( a_3 b_4 = a_4 \) and \( a_4 \neq 0 \), we have \( b_4 = 1 \), that is, \( a = (0, 0, 0, a_4 F), netu(a) = (0, 0, 0, F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \((0, 0, 0, a_4 F) \ast (c_1, c_2 T, c_3 I, c_4 F) = (0, 0, 0, F) \), we have \( a_4 (c_1 + c_2 + c_3 + c_4) = 1 \), so the opposite element of \( a \) should satisfy \( c_1 + c_2 + c_3 + c_4 = \frac{1}{a_4}, c_1 \in \mathbb{R} \).

Case A121: \( a_1 = a_2 = 0, a_3 \neq 0, a_3 + a_4 = 0 \), that is \( a = (0, 0, a_3 I, a_4 F), netu(a) = (0, 0, I, b_4 F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a \ast anti(a) = netu(a) \), that is \( (0, 0, a_3 I, a_4 F) \ast (c_1, c_2 T, c_3 I, c_4 F) = (0, 0, I, b_4 F) \). So we have \( a_3 (c_1 + c_2 + c_3 + c_4) = 1 \) and \( a_3 c_4 - a_3 (c_1 + c_2 + c_3 + c_4) = b_4 \) i.e., \( c_1 + c_2 + c_3 = \frac{1}{a_3} \) and \( b_4 = 1 \). Thus \( netu(a) = (0, 0, I, F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \), where \( c_1 + c_2 + c_3 = \frac{1}{a_3} \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A122: \( a_1 = a_2 = 0, a_3 \neq 0, a_3 + a_4 \neq 0 \). From \( (a_3 + a_4) b_4 = 0 \), we have \( b_4 = 0 \). that is \( a = (0, 0, a_3 I, a_4 F), netu(a) = (0, 0, I, 0), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a \ast anti(a) = netu(a) \), that is \( (0, 0, a_3 I, a_4 F) \ast (c_1, c_2 T, c_3 I, c_4 F) = (0, 0, I, 0) \). So we have \( a_3 (c_1 + c_2 + c_3) = 1 \) and \( a_3 c_4 - a_3 (c_1 + c_2 + c_3 + c_4) = 0 \) i.e., \( c_1 + c_2 + c_3 = \frac{1}{a_3} \) and \( c_4 = -\frac{a_4}{a_3(a_3+a_4)}. \) Thus \( netu(a) = (0, 0, I, 0), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \), where \( c_1 + c_2 + c_3 = \frac{1}{a_3}, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A2: when \( a_1 = 0, a_2 \neq 0 \). From \( a_2 b_2 = a_2 \), we have \( b_2 = 1 \), that is, \( a = (0, 0, a_3 I, a_4 F), netu(a) = (0, T, b_3 I, b_4 F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). In the same way, from \( a \ast netu(a) = a \), we have \( a_2 + a_3 = 0 \), so we discuss from \( a_2 = 0 \) or \( a_2 \neq 0 \).

Case A21: when \( a_1 = 0, a_2 \neq 0, a_2 + a_3 = 0 \). that is, \( a = (0, a_2 T, -a_2 I, a_4 F), netu(a) = (0, T, b_3 I, b_4 F), anti(a) = (c_1, c_2 T, c_3 I, c_4 F) \). In the same way, from \( a \ast netu(a) = a \), we have \( a_4 + a_4 (b_3 + b_4) = a_4 \), that is \( a_4 (b_3 + b_4) = 0 \), so we discuss from \( a_4 = 0 \) or \( a_4 \neq 0 \).
Case A211: when \( a_1 = 0, a_2 \neq 0, a_2 + a_3 = 0, a_4 = 0 \). That is, \( a = (0, a_2 T, -a_2 I, 0), \text{neut}(a) = (0, T, b_3, b_4 T, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( (0, a_2 T, -a_2 I, 0) * (c_1, c_2 T, c_3 I, c_4 F) = (0, T, b_3, b_4 F) \), we have \( a_2 (c_1 + c_2) = 1 \) and \( -a_2 (c_1 + c_2) = b_3 \), that is \( b_3 = -1 \). In the same way, we can get \( b_4 = 0 \). Thus \( \text{neut}(a) = (0, T, -I, 0) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), where \( c_1 + c_2 = \frac{1}{a_2} \), \( c_3, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A212: when \( a_1 = 0, a_2 \neq 0, a_2 + a_3 = 0, a_4 \neq 0 \). From \( a_4 (b_3 + b_4) = 0 \), we have \( b_3 + b_4 = 0 \), that is, \( a = (0, a_2 T, -a_2 I, a_4 F) \), \( \text{neut}(a) = (0, T, b_3 I, -b_3 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( (0, a_2 T, -a_2 I, a_4) * (c_1, c_2 T, c_3 I, c_4 F) = (0, T, b_3 I, -b_3 F) \), we have \( a_2 (c_1 + c_2) = 1 \) and \( -a_2 (c_1 + c_2) = b_3 \), i.e., \( b_3 = -1 \). Thus \( \text{neut}(a) = (0, T, -I, F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), where \( c_1 + c_2 = \frac{1}{a_2} \), \( c_3, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A222: when \( a_1 = 0, a_2 \neq 0, a_2 + a_3 \neq 0, a_2 + a_3 + a_4 \neq 0 \). From \( (a_2 + a_3) b_3 = 0 \), we have \( b_3 = 0 \). That is, \( a = (0, a_2 T, -a_2 I, a_4 F) \), \( \text{neut}(a) = (0, T, 0, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( (0, a_2 T, a_3 I, a_4 F) * (c_1, c_2 T, c_3 I, c_4 F) = (0, T, 0, b_4 F) \), we have \( a_2 (c_1 + c_2) = 1 \), \( c_3 = -\frac{a_3}{a_2 (a_2 + a_3)} \), \( a_4 + a_4 b_4 (c_1 + c_2 + c_3) = b_4 \), so we have \( b_4 = -1 \). Thus \( \text{neut}(a) = (0, T, 0, -F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), where \( c_1 + c_2 = \frac{1}{a_2} \), \( c_3, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case A222: when \( a_1 = 0, a_2 \neq 0, a_2 + a_3 \neq 0, a_2 + a_3 + a_4 \neq 0 \). From \( (a_2 + a_3 + a_4) b_4 = 0 \), we have \( b_4 = 0 \). That is, \( a = (0, a_2 T, a_3 I, a_4 F) \), \( \text{neut}(a) = (0, T, 0, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( (0, a_2 T, a_3 I, a_4 F) * (c_1, c_2 T, c_3 I, c_4 F) = (0, T, 0, b_4 F) \), we have \( a_2 (c_1 + c_2) = 1 \), \( c_3 = -\frac{a_3}{a_2 (a_2 + a_3)} \), \( a_4 + a_4 b_4 (c_1 + c_2 + c_3) = b_4 \), so we have \( b_4 = -1 \). Thus \( \text{neut}(a) = (0, T, 0, -F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), \( c_1 + c_2 = \frac{1}{a_2} \), \( c_3, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case B: when \( a_1 \neq 0 \).

In this case, \( a_1 b_1 = a_1 \) and \( a_1 \neq 0 \), we have \( b_1 = 1 \). That is \( a = (a_1, a_2 T, a_3 I, a_4 F) \), \( \text{neut}(a) = (1, b_2 T, b_3 I, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From Definition 1, \( a * \text{neut}(a) = a, \) that is \( a_1 b_2 + a_2 b_2 = a_2 \), so \( a_1 b_2, 2b_2 = a_2 \). So we discuss from \( a_1 + a_2 = 0 \) or \( a_1 + a_2 \neq 0 \).

Case B1: when \( a_1 \neq 0, a_1 + a_2 = 0 \). That is \( a = (a_1, -a_1 T, a_3 I, a_4 F) \), \( \text{neut}(a) = (1, -T, b_3 I, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a * \text{anti}(a) = \text{neut}(a) \), \( c_1 = \frac{1}{a_1} \), \( a_1 c_1 - a_1 c_2 - a_1 c_2 = b_2 \), so \( b_2 = -1 \). From \( a * \text{neut}(a) = a, \) so we have \( a_3 + a_3 b_2 + a_3 b_3 = a_3 \), i.e., \( a_3 (b_2 + b_3) = 0 \). So we discuss from \( a_3 = 0 \) or \( a_3 \neq 0 \).

Case B11: when \( a_1 \neq 0, a_1 + a_2 = 0, a_3 = 0 \). That is \( a = (a_1, -a_1 T, 0, a_4 F) \), \( \text{neut}(a) = (1, -T, b_3 I, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a * \text{anti}(a) = \text{neut}(a) \), i.e., \( (a_1, -a_1 T, 0, a_4 F) * (c_1, c_2 T, c_3 I, c_4 F) = (1, -T, b_3 I, b_4 F) \), we have \( c_1 = \frac{1}{a_1} \), \( b_3 = 0, b_4 = 1 \). Thus \( \text{neut}(a) = (1, -T, 0, 0) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), \( c_1 = \frac{1}{a_1} \) and \( c_2, c_3, c_4 \) can be chosen arbitrarily in \( \mathbb{R} \).

Case B12: when \( a_1 \neq 0, a_1 + a_2 = 0, a_3 \neq 0 \). From \( a_3 (b_2 + b_3) = 0 \), we have \( b_2 + b_3 = 0 \), i.e., \( b_3 = 1 \). That is \( a = (a_1, -a_1 T, a_3 I, a_4 F) \), \( \text{neut}(a) = (1, -T, b_3 I, b_4 F) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \). From \( a * \text{anti}(a) = \text{neut}(a) \), i.e., \( (a_1, -a_1 T, 0, a_4 F) * (c_1, c_2 T, c_3 I, c_4 F) = (1, -T, b_3 I, b_4 F) \), we have \( c_1 = \frac{1}{a_1} \), \( b_3 = 0, b_4 = 1 \). Thus \( \text{neut}(a) = (1, -T, 0, 0) \), \( \text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F) \), \( c_1 = \frac{1}{a_1} \) and \( c_2, c_3 + c_4 = \frac{1}{a_1} \).
(a_1, -a_1T, a_3I, -a_3F) \ast (c_1, c_2T, c_3I, c_4F) = (1, -T, I, b_4F), \text{ we have } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, -a_2(c_1 + c_2 + c_3) = b_4, \text{ i.e., } b_4 = -1. \text{ Thus } neu(a) = (1, -T, I, -F), anti(a) = (c_1, c_2T, c_3I, c_4F), \text{ where } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, \frac{c_4}{a_3(a_3 + a_4)}. \text{ Thus } neu(a) = (1, -T, I, -F), anti(a) = (c_1, c_2T, c_3I, c_4F), \text{ where } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, \frac{c_4}{a_3(a_3 + a_4)}.

Case B22: when a_1 \neq 0, a_1 + a_2 \neq 0, a_3 \neq 0, a_3 + a_4 \neq 0, \text{ from } (a_3 + a_4)b_4 = 0, \text{ we have } b_4 = 0. \text{ That is } a = (a_1, -a_1T, a_3I, a_4F), \text{ neu}(a) = (1, -T, I, 0), \text{ anti}(a) = (c_1, c_2T, c_3I, c_4F). \text{ From } a \ast anti(a) = neu(a), \text{ i.e., } (a_1, -a_1T, a_3I, a_4F) \ast (c_1, c_2T, c_3I, c_4F) = (1, -T, I, 0), \text{ we have } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, c_4 = \frac{a_1}{a_3(a_3 + a_4)}. \text{ Thus } neu(a) = (1, -T, I, -F), anti(a) = (c_1, c_2T, c_3I, c_4F), \text{ where } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, \frac{c_4}{a_3(a_3 + a_4)}.

Case B22: when a_1 \neq 0, a_1 + a_2 \neq 0, a_3 \neq 0, a_3 + a_4 \neq 0, \text{ from } (a_1 + a_2)b_4 = 0, \text{ we have } b_4 = 0. \text{ That is } a = (a_1, a_2T, a_3I, a_4F), \text{ neu}(a) = (1, 0, b_3I, b_4F), \text{ anti}(a) = (c_1, c_2T, c_3I, c_4F). \text{ From } a \ast anti(a) = neu(a), \text{ i.e., } (a_1, a_2T, a_3I, a_4F) \ast (c_1, c_2T, c_3I, c_4F) = (1, 0, b_3I, b_4F), \text{ we have } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, c_4 = \frac{a_1}{a_3(a_3 + a_4)}. \text{ Thus } neu(a) = (1, 0, -T, 0), \text{ anti}(a) = (c_1, c_2T, c_3I, c_4F), \text{ where } c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_5}, \frac{c_4}{a_3(a_3 + a_4)}.

Finally, we should show that all the above cases include each element } a \in \mathbb{N}, \text{ i.e., } a_i, i = 1, 2, 3, 4 \text{ can take all the values on } \mathbb{R}. \text{ It is obvious that } a_1 \text{ can take all the values on } \mathbb{R} \text{ because } a_1 = 0 \text{ according to case A and that } a_1 \neq 0 \text{ according to case B. } \text{Moreover, for case A, } a_2 \text{ can take all the values on } \mathbb{R} \text{ because case A1 according to } a_2 = 0 \text{ and case A2 according to } a_2 \neq 0. \text{ For case B, } a_3 \text{ can take all the values on } \mathbb{R} \text{ because case B1 according to } a_1 + a_2 = 0 \text{ and case B2 according to } a_1 + a_2 \neq 0. \text{ That is for each element } a = (a_1, a_2, a_3, a_4) \in \mathbb{N}, a_1, a_2 \text{ can select all of value in } \mathbb{R}. \text{ We will verify that } a_3 \text{ and } a_4 \text{ can take all the values on } \mathbb{R} \text{ when case A1 or A2 or B1 or B2 respectively.}
For case A1, \( a_3 \) can take all the value in \( \mathbb{R} \) because case A11 according to \( a_3 = 0 \) and case A12 according to \( a_3 \neq 0 \). Similarly, for case A1, \( a_4 \) can take all the value in \( \mathbb{R} \) because case A11 according to \( a_4 = 0 \) and case A12 according to \( a_4 \neq 0 \). For case A12, \( a_4 \) can take all the value in \( \mathbb{R} \) because case A121 according to \( a_4 = 0 \) and case A122 according to \( a_4 \neq 0 \). The unique \( \Box \) point represents the case A111, the + points represent the case A112, the * points represent the case A121 and the • points represent the case A122. This explain the that for case A1, \( a_3 \) and \( a_4 \) can take all the points on the plane. For case A2, B1 or B2, we can get that \( a_3 \) and \( a_4 \) can take all the points on the plane respectively. The top right subgraph of Figure 1 represents the case A2 if we select \( a_1 = 0, a_2 = 1 \), the bottom left subgraph of Figure 1 represents the case B1 if we select \( a_1 = 1, a_2 = -1 \) and bottom right subgraph of Figure 1 represents the case B2 if we select \( a_1 = 1, a_2 = 0 \). The figure intuitively illustrates that all the points \((a_1, a_2, a_3, a_4)\), \(a_i \in \mathbb{R}\) are included.

Through the above analysis, we can get that for each element \( a \in NQ \), there exists the neutral element \( \text{neut}(a) \) and opposite element \( \text{anti}(a) \).

![Figure 1](image-url)

**Figure 1.** The demonstration figure shows that case A1 \((a_1 = a_2 = 0, \) the top left subgraph) or A2 (select \( a_1 = 0, 0 \neq a_2 = 1, \) the top right subgraph) or B1 (Select \( a_1 \neq 0, a_2 = -1 \) which means \( a_1 + a_2 = 0, \) the bottom left subgraph) or B2 (select \( a_1 = 1, a_2 = 0, \) which means \( a_1 + a_2 \neq 0, \) the bottom right subgraph) can take all the values on the plane.

For algebra system \((NQ, *)\), Table 1 gives all the subset, which has the same neutral element, and the corresponding neutral element and opposite elements.
The Subset of $NQ$ | Neutral Elements | Opposite Element $(c_1, c_2 T, c_3 I, c_4 F)$
--- | --- | ---
$\{0, 0, 0, 0\}$ | $(0, 0, 0, 0)$ | $c_i \in \mathbb{R}$
$\{(0, 0, 0, a_4 F) | a_4 \neq 0\}$ | $(0, 0, 0, F)$ | $c_1 + c_2 + c_3 + c_4 = \frac{1}{a_4}$
$\{(0, 0, a_3 I, -a_3 F) | a_3 \neq 0\}$ | $(0, 0, I, -F)$ | $c_1 + c_2 + c_3 = \frac{1}{a_3}, c_4 \in \mathbb{R}$
$\{(0, 0, a_3 I, a_4 F) | a_3 \neq 0, a_3 + a_4 \neq 0\}$ | $(0, 0, I, 0)$ | $c_1 + c_2 + c_3 = \frac{1}{a_3}, c_4 = -\frac{a_3}{a_3(a_3 + a_4)}$
$\{(0, a_2 T, -a_2 I, 0) | a_2 \neq 0\}$ | $(0, T, -I, 0)$ | $c_1 + c_2 = \frac{1}{a_2}, c_3, c_4 \in \mathbb{R}$
$\{(0, a_2 T, -a_2 I, a_4 F) | a_2 \neq 0, a_4 \neq 0\}$ | $(0, T, -I, F)$ | $c_1 + c_2 = \frac{1}{a_2}, c_3 + c_4 = \frac{1}{a_2} - \frac{a_2}{a_2}$
$\{(0, a_2 T, a_3 I, a_4 F) | a_2 \neq 0, a_2 + a_3 \neq 0\}$ | $(0, T, 0, -F)$ | $c_1 + c_2 = \frac{1}{a_2}, c_3 = -\frac{a_2}{a_2(a_2 + a_3)}, c_4 \in \mathbb{R}$
$\{(0, a_2 T, a_3 I, a_4 F) | a_2 \neq 0, a_2 + a_3 \neq 0\}$ | $(0, T, 0, 0)$ | $c_1 + c_2 = \frac{1}{a_2}, c_3 = -\frac{a_2}{a_2(a_2 + a_3)}, c_4 = -\frac{a_2}{a_2[a_2 + a_3]}$
$\{(a_1, -a_1 T, 0, 0) | a_1 \neq 0\}$ | $(1, -T, 0, 0)$ | $c_1 = \frac{1}{a_1}, c_2, c_3, c_4 \in \mathbb{R}$
$\{(a_1, -a_1 T, 0, a_4 F) | a_1 \neq 0, a_4 \neq 0\}$ | $(1, -T, 0, 0)$ | $c_1 = \frac{1}{a_1}, c_2 + c_3 + c_4 = \frac{1}{a_4} - \frac{1}{a_1}$
$\{(a_1, -a_1 T, a_3 I, -a_3 F) | a_1 \neq 0, a_3 \neq 0\}$ | $(1, -T, I, -F)$ | $c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_1}, c_4 \in \mathbb{R}$
$\{(a_1, -a_1 T, a_3 I, a_4 F) | a_1 \neq 0, a_3 \neq 0\}$ | $(1, -T, I, 0)$ | $c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_1}, c_4 = -\frac{a_1}{a_1(a_1 + a_4)}$
$\{(a_1, a_2 T, a_3 I, 0) | a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 + a_4 = 0\}$ | $(1, 0, -I, 0)$ | $c_1 = \frac{1}{a_1}, c_2 = -\frac{a_2}{a_2[a_1 + a_2]}, c_3, c_4 \in \mathbb{R}$
$\{(a_1, a_2 T, a_3 I, a_4 F) | a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 + a_4 = 0\}$ | $(1, 0, 0, -F)$ | $c_1 = \frac{1}{a_1}, c_2 = -\frac{a_2}{a_2[a_1 + a_2]}, c_3 + c_4 = \frac{1}{a_2}$
$\{(a_1, a_2 T, a_3 I, a_4 F) | a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 + a_4 \neq 0\}$ | $(1, 0, 0, 0)$ | $c_1 = \frac{1}{a_1}, c_2 = -\frac{a_2}{a_2[a_1 + a_2]}, c_3 = \frac{a_3}{a_1 + a_2}, c_4 \in \mathbb{R}$

**Example 1.** For the algebra system $(NQ, \ast)$, if $a = (a_1, a_2, a_3, a_4) = (1, -T, 3I, -F)$, i.e., $a_1 \neq 0, a_1 + a_2 = 0, a_3 \neq 0, a_3 + a_4 \neq 0$, then from Table 1, we can get $\text{neut}(a) = (1, -T, I, 0)$. Let $\text{anti}(a) = (c_1, c_2 T, c_3 I, c_4 F)$, so $c_1 = \frac{1}{a_1} = 1, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_1} = -\frac{2}{a_1}, c_4 = -\frac{a_3}{a_1(a_3 + a_4)} = \frac{1}{a_1}$, so $\text{anti}(a) = (1, c_2 T, c_3 I, \frac{1}{a_1} F)$, where $c_2 + c_3 = -\frac{2}{a_1}$. Thus we can easily get the neutral element and opposite elements of each neutrosophic quadruple number. For more examples, see the following:

1. Let $b = (0, 0, I, -F)$, then $\text{neut}(b) = (0, 0, I, -F)$ and $\text{anti}(b) = (c_1, c_2 T, c_3 I, c_4 F)$, where $c_1 + c_2 + c_3 = 1, c_4$ can be chosen arbitrarily in $\mathbb{R}$.
2. Let $c = (1, T, I, -F)$, then $\text{neut}(c) = (1, 0, 0, 0)$ and $\text{anti}(c) = (1, -\frac{1}{2} T, -\frac{1}{2} I, \frac{1}{2} F)$.
3. Let $d = (0, T, I, F)$, then $\text{neut}(d) = (0, T, 0, 0)$ and $\text{anti}(d) = (c_1, c_2 T, -\frac{1}{2} I, -\frac{1}{2} F)$, where $c_1 + c_2 = 1$.

In the following, we will discuss the algebra structure properties of $(NQ, \ast)$.

**Proposition 3.** For algebra system $(NQ, \ast)$, let $NS = \{\text{neut}(a) | a \in NQ\}$, we have:

1. $NS = \{(1, 0, 0, 0), (0, 0, 0, F), (0, 0, I, -F), (0, 0, I, 0), (0, T, -I, 0), (0, T, -I, F), (0, T, 0, -F), (0, T, 0, 0), (1, -T, 0, 0), (1, -T, 0, F), (1, -T, I, -F), (1, -T, I, 0), (1, 0, -I, 0), (1, 0, -I, F), (1, 0, 0, -F), (1, 0, 0, 0)\}$.
2. $NS$ is closed with respect to operation $\ast$. 

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Table 1. The corresponding neutral element and opposite elements for $(NQ, \ast)$. 

(3) Set $IS = \{a|a^2 = a, a \in NQ\}$, which is all the set of idempotent elements of $(NQ, \ast)$, then $NS = IS$.

Proof. (1) Obviously.

(2) If $c,d \in NS$, that is $\text{neut}(a) = c, \text{neut}(b) = d, a, b \in NQ$. From Proposition 1, $\text{neut}(a) \ast \text{neut}(b) = \text{neut}(a \ast b)$, i.e., $c \ast d = \text{neut}(a \ast b)$, then form Theorem 1, every element in NQ has neutral element, so $a \ast b$ also has neutral element, that is $\text{neut}(a \ast b) \in NS$, i.e., $c \ast d \in NS$, thus $NS$ is closed with respect to operation $\ast$.

(3) From Proposition 1, $\text{neut}(a) \ast \text{neut}(a) = \text{neut}(a)$, so $\text{neut}(a)$ is a idempotent element and $NS \subseteq IS$. On the other hand if $a$ is a idempotent element, so $a \ast a = a$, that is $a$ exists the neutral element $a$ and the opposite element $a$, so $a$ is a neutral element, that is $IS \subseteq NS$. Thus $NS = IS$. \qquad \Box

Proposition 4. For algebra system $(NQ, \ast)$, let $V_c = \{a|a \in NQ \land \text{neut}(a) = c\}$, $V_{c \ast d} = \{a \ast b|a, b \in NQ \land \text{neut}(a) = c \land \text{neut}(b) = d\}$, we have:

1. $V_c$ is closed with respect to operation $\ast$.
2. $V_{c \ast d}$ is closed with respect to operation $\ast$.

Proof. (1) If $a, b \in V_c$, that is $\text{neut}(a) = \text{neut}(b) = c$. From Proposition 1, $\text{neut}(a) \ast \text{neut}(b) = \text{neut}(a \ast b)$, we can see that $\text{neut}(a \ast b) = \text{neut}(a) = c$, i.e., the neutral element of $a \ast b$ is the neutral element of $a$, so $a \ast b \in V_c$, that is $V_c$ is closed with respect to operation $\ast$.

(2) If $a_1 \ast b_1, a_2 \ast b_2 \in V_{c \ast d}$, i.e., $\text{neut}(a_1) = \text{neut}(a_2) = c, \text{neut}(b_1) = \text{neut}(b_2) = d$. From Proposition 3(2), $a_1 \ast a_2 = a_3 \in V_c, b_1 \ast b_2 = b_3 \in V_d$, so $\text{neut}(a_3) = c, \text{neut}(b_3) = d$, from $(a_1 \ast b_1) \ast (a_2 \ast b_2) = a_3 \ast b_3$, so $\text{neut}(a_1 \ast b_1) \ast (a_2 \ast b_2) = \text{neut}(a_3 \ast b_4)$, that is $a_3 \ast b_4 \in V_{c \ast d}$, that means $a_1 \ast a_2 \ast b_1 \ast b_2 \in V_{c \ast d}$. Thus $V_{c \ast d}$ is closed with respect to operation $\ast$. \qquad \Box

Definition 7. Assume that $(N, \ast)$ is a neutrosophic triplet group and $H$ be a nonempty subset of $N$. Then $H$ is called a neutrosophic triplet subgroup of $N$ if:

1. $a \ast b \in H$ for all $a, b \in H$;

2. there exists $\text{anti}(a) \in \{\text{anti}(a)\}$ such that $\text{anti}(a) \in H$ for all $a \in H$, where $\{\text{anti}(a)\}$ is the set of opposite element of $a$ in $(N, \ast)$.

Theorem 2. For algebra system $(NQ, \ast)$, let $V_c = \{a|a \in NQ \land \text{neut}(a) = c\}$, $V_{c \ast d} = \{a \ast b|a, b \in NQ \land \text{neut}(a) = c, \text{neut}(b) = d\}$, we have:

1. $V_c$ is a neutrosophic triplet subgroup of $NQ$.
2. $V_{c \ast d}$ is a neutrosophic triplet subgroup of $NQ$.

Proof. (1) From Proposition 3, we can see that $V_c$ is closed with respect to operation $\ast$. In the following, we will prove there exists $\text{anti}(a) \in \{\text{anti}(a)\}$ such that $\text{anti}(a) \in V_c$ for all $a \in V_c$.

Proof by contradiction.

Assume that $\{\text{anti}(a)\} \cap V_c = \emptyset$. From Proposition 1 we can see that $a \ast \text{anti}(a) = c$. On the other hand, $\text{anti}(a) \in NQ$, so $\text{anti}(a)$ exists neutral element, denoted by $\text{neut}(\text{anti}(a))$. Being $\text{anti}(a) \notin V_c$, so $\text{neut}(\text{anti}(a)) \neq c$.

From $a \ast \text{anti}(a) = c$, we have $a \ast \text{anti}(a) \ast \text{neut}(\text{anti}(a)) = c \ast \text{neut}(\text{anti}(a))$, being $\text{anti}(a) \ast \text{neut}(\text{anti}(a)) = \text{anti}(a)$ and $a \ast \text{anti}(a) = c$, we have $c \ast \text{neut}(\text{anti}(a)) = c$, and then we have $a \ast c \ast \text{neut}(\text{anti}(a)) = a \ast c = a$, that means $a \ast \text{neut}(\text{anti}(a)) = a$, so $\text{neut}(\text{anti}(a))$ is also a neutral element of $a$. This leads to the contradiction being the uniqueness of neutral element for each element. Therefore $\{\text{anti}(a)\} \cap V_c \neq \emptyset$. Thus from Definition 7, $V_c$ is a neutrosophic triplet subgroup of $NQ$.

(2) The same way we can get $V_{c \ast d}$ is a neutrosophic triplet subgroup of $NQ$. \qquad \Box

Theorem 3. For algebra system $(NQ, \ast)$, let $V_c = \{a|a \in NQ \land \text{neut}(a) = c\}$, we have:

1. $V_c \cap V_d = \emptyset$ if $c \neq d$. 


(2) \( NQ = \cup_{c \in NS} V_c \). So \( \cup_{c \in NS} V_c \) is a partition of \( NQ \), where \( NS \) is a set, which contains all the neutral elements of \( (NQ, \ast) \).

**Proof.** 1) Proof by contradiction.

Assume \( V_c \cap V_d \neq \emptyset \) when \( c \neq d \), so exist \( a \in V_c \cap V_d \) such that \( a \) has two neutral elements \( c \) and \( d \). This leads to the contradiction being the uniqueness of neutral element. So \( V_c \cap V_d = \emptyset \) if \( c \neq d \).

2) From the proof of Theorem 1, we can get \( NQ = \cup_{c \in NS} V_c \). So \( \cup_{c \in NS} V_c \) is a partition of \( NQ \). □

For the algebra system \((NQ, \ast)\), we have the similar results. We describe as following and omit the proof.

**Theorem 4.** For the algebra system \((NQ, \ast)\), for every element \( a \in NQ \), there exists the neutral element \( \text{neut}(a) \) and opposite element \( \text{anti}(a) \).

For algebra system \((NQ, \ast)\), Table 2 gives all the subset, which has the same neutral element, and the corresponding neutral element and opposite elements.

**Table 2.** The corresponding neutral element and opposite elements for \((NQ, \ast)\).

<table>
<thead>
<tr>
<th>The Subset of NQ</th>
<th>Neutral Elements</th>
<th>Opposite Element ((c_1, c_2 T, c_3 I, c_4 F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0,0,0,0])</td>
<td>((0,0,0,0))</td>
<td>(c_1 \in R)</td>
</tr>
<tr>
<td>([0, a_2 T,0,0]</td>
<td>a_2 \neq 0)</td>
<td>((0, T,0,0)) (c_1 + c_2 + c_3 + c_4 = \frac{1}{a_2})</td>
</tr>
<tr>
<td>([0, -a_3 T,a_3 I,0]</td>
<td>a_3 \neq 0)</td>
<td>((0,-T,I,0)) (c_1 + c_3 + c_4 = \frac{1}{a_3}, c_2 \in R)</td>
</tr>
<tr>
<td>([0,a_2 T,a_3 I,a_4 F]</td>
<td>a_3 \neq 0, a_2 + a_3 \neq 0)</td>
<td>((0,0,I,0)) (c_1 + c_3 + c_4 = \frac{1}{a_3}, c_2 = -\frac{a_2}{\frac{a_1 a_2 + a_3}{a_2}})</td>
</tr>
<tr>
<td>([0,0,-a_4 I,a_4 F]</td>
<td>a_4 \neq 0)</td>
<td>((0,0,-I,F)) (c_1 + c_4 = \frac{1}{a_4}, c_2, c_3 \in R)</td>
</tr>
<tr>
<td>([0,a_2 T,-a_4 I,a_4 F]</td>
<td>a_2 \neq 0, a_4 \neq 0)</td>
<td>((0,T,-I,F)) (c_1 + c_4 = \frac{1}{a_2}, c_2 + c_3 = \frac{1}{a_4} - \frac{1}{a_2})</td>
</tr>
<tr>
<td>([0,a_2 T,a_3 I,a_4 F]</td>
<td>a_4 \neq 0, a_3 + a_4 \neq 0, a_2 + a_3 + a_4 = 0)</td>
<td>((0,-T,0,F)) (c_1 + c_4 = \frac{1}{a_2}, c_3 = -\frac{a_3}{a_1 (a_2 + a_3)}, c_2 \in R)</td>
</tr>
<tr>
<td>([0,a_2 T,a_3 I,a_4 F]</td>
<td>a_4 \neq 0, a_3 + a_4 \neq 0, a_2 + a_3 + a_4 \neq 0)</td>
<td>((0,0,0,F)) (c_1 + c_4 = \frac{1}{a_2}, c_3 = -\frac{a_3}{a_1 (a_2 + a_3)}, c_2 = -\frac{a_2}{a_1 (a_2 + a_3)})</td>
</tr>
<tr>
<td>([a_1,0,0,-a_1 F]</td>
<td>a_1 \neq 0)</td>
<td>((1,0,0,-F)) (c_1 = -\frac{a_1}{a_2}, c_2, c_3, c_4 \in R)</td>
</tr>
<tr>
<td>([a_1, a_2 T,0,-a_1 F]</td>
<td>a_1 \neq 0, a_2 \neq 0)</td>
<td>((1,T,0,-F)) (c_1 = \frac{1}{a_2}, c_2 + c_3 + c_4 = \frac{1}{a_2} - \frac{1}{a_1})</td>
</tr>
<tr>
<td>([a_1,-a_3 T,a_3 I,-a_1 F]</td>
<td>a_1 \neq 0, a_3 \neq 0)</td>
<td>((1,-T,I,-F)) (c_1 = \frac{1}{a_2}, c_3 + c_4 = \frac{1}{a_3} - \frac{1}{a_1}, c_2 \in R)</td>
</tr>
<tr>
<td>([a_1,a_2 T,a_3 I,-a_1 F]</td>
<td>a_1 \neq 0, a_2 + a_3 = 0)</td>
<td>((1,0,I,-F)) (c_1 = \frac{1}{a_2}, c_3 + c_4 = \frac{1}{a_3} - \frac{1}{a_1}, c_2 = -\frac{a_2}{a_1 (a_2 + a_3)})</td>
</tr>
<tr>
<td>([a_1,0,a_3 I,a_4 F]</td>
<td>a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 = 0)</td>
<td>((1,0,-I,0)) (c_1 = \frac{1}{a_2}, c_4 = -\frac{a_1}{a_3 (a_1 + a_4)}, c_2, c_3 \in R)</td>
</tr>
<tr>
<td>([a_1,a_2 T,a_3 I,a_4 F]</td>
<td>a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 \neq 0)</td>
<td>((1,T,-I,0)) (c_1 = \frac{1}{a_2}, c_4 = -\frac{a_1}{a_3 (a_1 + a_4)}, c_2 + c_3 = \frac{1}{a_2} - \frac{1}{a_1})</td>
</tr>
<tr>
<td>([a_1,a_2 T,a_3 I,a_4 F]</td>
<td>a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 \neq 0, a_2 + a_3 + a_4 = 0)</td>
<td>((1,-T,0,0)) (c_1 = \frac{1}{a_2}, c_4 = -\frac{a_1}{a_3 (a_1 + a_4)}, c_3 = \frac{1}{a_2} - \frac{1}{a_1})</td>
</tr>
<tr>
<td>([a_1,a_2 T,a_3 I,a_4 F]</td>
<td>a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 \neq 0)</td>
<td>((1,0,0,0)) (c_1 = \frac{1}{a_2}, c_4 = -\frac{a_1}{a_3 (a_1 + a_4)}, c_2 = -\frac{a_2}{a_1 (a_2 + a_3 + a_4)})</td>
</tr>
</tbody>
</table>
Theorem 5. For an algebra system \((NQ, \ast)\), let \(V_c = \{a|a \in NQ \land \text{neut}(a) = c\}, V_{c+d} = \{a \ast b|a, b \in NQ \land \text{neut}(a) = c, \text{neut}(b) = d\}\), we have:

1. \(V_c\) is a neutrosophic triplet subgroup of NQ.
2. \(V_{c+d}\) is a neutrosophic triplet subgroup of NQ.

Theorem 6. For algebra system \((NQ, \ast)\), let \(V_c = \{a|a \in NQ \land \text{neut}(a) = c\},\) we have:

1. \(V_c \cap V_d = \emptyset\) if \(c \neq d\).
2. \(NQ = \bigcup_{c \in NS} V_c\). So \(\bigcup_{c \in NS} V_c\) is a partition of NQ, where NS is a set, which contains all the neutral elements of \((NQ, \ast)\).

4. Two Kinds of Degenerate Systems of Neutrosophic Quadruple Numbers

The neutrosophic quadruple numbers consider \((T, I, F)\) to solve real problems. In this section, we will explore two kinds of degenerate systems about neutrosophic quadruple numbers. The first system is only consider logical true, and the second system is only consider logical true and logical indeterminacy.

4.1. The Neutrosophic Binary Numbers

Definition 8. A neutrosophic binary number is a number of the form \((a, bT)\), where \(T\) have their usual neutrosophic logic true and \(a, b \in \mathbb{R}\) or \(\mathbb{C}\). The set NB defined by

\[
NB = \{(a, bT) : a, b \in \mathbb{R} \text{ or } \mathbb{C}\}.
\]

is called a neutrosophic set of binary numbers. For a neutrosophic binary number \((a, bT)\), \(a\) is called the known part and \((bT)\) is called the unknown part.

Definition 9. Let \(a = (a_1, a_2 T)\), \(b = (b_1, b_2 T) \in NB\), the multiplication operation is defined as following:

\[
a \ast b = (a_1, a_2 T) \ast (b_1, b_2 T) = (a_1 b_1, (a_1 b_2 + a_2 b_1 + a_2 b_2) T).
\]

We have the following results similar to \((NQ, \ast)\).

Theorem 7. For the algebra system \((NB, \ast)\), for every element \(a \in NB\), there exists the neutral element \(\text{neut}(a)\) and opposite element \(\text{anti}(a)\).

For algebra system \((NB, \ast)\), Table 3 gives all the subset, which has the same neutral element, and the corresponding neutral element and opposite elements.

<table>
<thead>
<tr>
<th>The Subset</th>
<th>Neutral Elements</th>
<th>Opposite Element ((c_1, c_2 T))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{(0, 0)}</td>
<td>((0, 0))</td>
<td>(c_i \in \mathbb{R})</td>
</tr>
<tr>
<td>{(0, a_2 T)</td>
<td>a_2 \neq 0}</td>
<td>((0, T))</td>
</tr>
<tr>
<td>{(a_1, -a_1 T)</td>
<td>a_1 \neq 0}</td>
<td>((1, 0))</td>
</tr>
<tr>
<td>{(a_1, a_2 T)</td>
<td>a_1 \neq 0}</td>
<td>((1, -T))</td>
</tr>
</tbody>
</table>

Theorem 8. For algebra system \((NB, \ast)\), let \(V_c = \{a|a \in NB \land \text{neut}(a) = c\}, V_{c+d} = \{a \ast b|a, b \in NB \land \text{neut}(a) = c, \text{neut}(b) = d\}\), we have:

1. \(V_c\) is a neutrosophic triplet subgroup of NB.
2. \(V_{c+d}\) is a neutrosophic triplet subgroup of NB.
Theorem 9. For an algebra system \((NB, *)\), let \(V_c = \{a|a \in NB \land \text{neut}(a) = c\}\), we have:

1. \(V_c \cap V_d = \emptyset\) if \(c \neq d\).
2. \(NB = \cup_{c \in NS} V_c\). So \(\cup_{c \in NS} V_c\) is a partition of \(NB\), where \(NS\) is a set, which contains all the neutral elements of \((NB, *)\).

4.2. The Neutrosophic Triple Numbers

Definition 10. A neutrosophic triple number is a number of the form \((a, b, c)\), where \(T, I\) have their usual neutrosophic logic meanings and \(a, b, c \in \mathbb{R}\) or \(\mathbb{C}\). The set \(NT\) defined by

\[
NT = \{(a, b, c) : a, b, c \in \mathbb{R} \text{ or } \mathbb{C}\}
\]

is called a neutrosophic set of triple numbers. For a neutrosophic triple number \((a, b, c)\), \(a\) is called the known part and \((b, c)\) is called the unknown part.

Definition 11. Let \(a = (a_1, a_2 T, a_3 I), b = (b_1, b_2 T, b_3 I) \in NT\), suppose in an pessimistic way, the neutrosophic expert considers the prevalence order \(T \prec I\). Then the multiplication operation is defined as following:

\[
a * b = (a_1, a_2 T, a_3 I) * (b_1, b_2 T, b_3 I) = (a_1 b_1, (a_1 b_2 + a_2 b_1 + a_2 b_2) T, (a_1 b_3 + a_2 b_3 + a_3 b_1 + a_3 b_2 + a_3 b_3) I).
\]

Suppose in an optimistic way the neutrosophic expert considers the prevalence order \(T \succ I\). Then:

\[
a * b = (a_1, a_2 T, a_3 I) * (b_1, b_2 T, b_3 I) = (a_1 b_1, (a_1 b_2 + a_2 b_1 + a_2 b_2 + a_2 b_3) T, (a_1 b_3 + a_3 b_1 + a_3 b_3) I).
\]

Theorem 10. For the algebra system \((NT, *)\), for every element \(a \in NT\), there exists the neutral element \(\text{neut}(a)\) and opposite element \(\text{anti}(a)\).

For algebra system \((NT, *)\), Table 4 gives all the subset, which has the same neutral element, and the corresponding neutral element and opposite elements.

<table>
<thead>
<tr>
<th>The Subset</th>
<th>Neutral Elements</th>
<th>Opposite Element ((c_1, c_2 T, c_3 I))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0,0,0}</td>
<td>(0,0,0)</td>
<td>(c_1, c_2, c_3 \in \mathbb{R})</td>
</tr>
<tr>
<td>{(0,0,a_3 I)</td>
<td>a_3 \neq 0}</td>
<td>(0,0,1) (c_1 + c_2 + c_3 = \frac{1}{a_3})</td>
</tr>
<tr>
<td>{(0,a_2 T,−a_2 I)</td>
<td>a_2 \neq 0, a_2 + a_3 = 0}</td>
<td>(0,T,−1) (c_1 + c_2 = \frac{1}{a_2}, c_3 \in \mathbb{R})</td>
</tr>
<tr>
<td>{(0,a_2 T, a_3 I)</td>
<td>a_2 \neq 0, a_2 + a_3 \neq 0}</td>
<td>(0,T,0) (c_1 + c_2 = \frac{1}{a_2}, c_3 = -\frac{a_3}{a_2(a_2 + a_3)})</td>
</tr>
<tr>
<td>{(a_1,−a_1 T,0)</td>
<td>a_1 \neq 0}</td>
<td>(1,−T,0) (c_1 = \frac{1}{a_1}, c_2, c_3 \in \mathbb{R})</td>
</tr>
<tr>
<td>{(a_1,−a_1 T, a_3 I)</td>
<td>a_1 \neq 0, a_3 \neq 0}</td>
<td>(1,−T,1) (c_1 = \frac{1}{a_1}, c_2 + c_3 = \frac{1}{a_3} - \frac{1}{a_1})</td>
</tr>
<tr>
<td>{(a_1, a_2 T, a_3 I)</td>
<td>a_1 \neq 0, a_1 + a_2 + a_3 \neq 0, a_2 + a_3 = 0}</td>
<td>(1,0,−I) (c_1 = \frac{1}{a_1}, c_2 = -\frac{a_2}{a_1(a_1 + a_2)}, c_3 \in \mathbb{R})</td>
</tr>
<tr>
<td>{(a_1, a_2 T, a_3 I)</td>
<td>a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 \neq 0}</td>
<td>(1,0,0) (c_1 = \frac{1}{a_1}, c_2 = -\frac{a_2}{a_1(a_1 + a_2)}, c_3 = -\frac{a_3}{(a_1 + a_2)(a_1 + a_2 + a_3)})</td>
</tr>
</tbody>
</table>
Theorem 11. For an algebra system $\langle NT, \ast \rangle$, let $V_c = \{a| a \in NT \land \text{neut}(a) = c\}$, $V_{c \oplus d} = \{a \ast b| a, b \in NT \land \text{neut}(a) = c, \text{neut}(b) = d\}$, we have:

1. $V_c$ is a neutrosophic triplet subgroup of $NT$.
2. $V_{c \oplus d}$ is a neutrosophic triplet subgroup of $NT$.

Theorem 12. For an algebra system $\langle NT, \ast \rangle$, let $V_c = \{a| a \in NT \land \text{neut}(a) = c\}$, we have:

1. $V_c \cap V_d = \emptyset$ if $c \neq d$.
2. $NT = \bigcup_{c \in NS} V_c$. So $\bigcup_{c \in NS} V_c$ is a partition of $NT$, where $NS$ is a set, which contains all the neutral elements of $\langle NT, \ast \rangle$.

Theorem 13. For the algebra system $\langle NT, \ast \rangle$, for every element $a \in NT$, there exists the neutral element $\text{neut}(a)$ and opposite element $\text{anti}(a)$.

For an algebra system $\langle NT, \ast \rangle$, Table 5 gives all the subset, which has the same neutral element, and the corresponding neutral element and opposite elements.

<table>
<thead>
<tr>
<th>The Subset</th>
<th>Neutral Elements</th>
<th>Opposite Element $(c_1, c_2T, c_3I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(0,0,0)}$</td>
<td>$(0,0,0)$</td>
<td>$c_1 \in \mathbb{R}$</td>
</tr>
<tr>
<td>${(0,a_2T,0)</td>
<td>a_2 \neq 0}$</td>
<td>$(0,T,0)$</td>
</tr>
<tr>
<td>${(0,a_3T,-a_1I)</td>
<td>a_3 \neq 0, a_2 + a_3 = 0}$</td>
<td>$(0,-T, I)$</td>
</tr>
<tr>
<td>${(0,a_2T,a_3I)</td>
<td>a_3 \neq 0, a_2 + a_3 \neq 0}$</td>
<td>$(0,0, I)$</td>
</tr>
<tr>
<td>${(a_1,0, -a_1I)</td>
<td>a_1 \neq 0}$</td>
<td>$(1,0, -I)$</td>
</tr>
<tr>
<td>${(a_1,a_2T,-a_1I)</td>
<td>a_1 \neq 0, a_2 \neq 0}$</td>
<td>$(1,T, -I)$</td>
</tr>
<tr>
<td>${(a_1,a_2T,a_3I)</td>
<td>a_1 \neq 0, a_1 + a_3 \neq 0, a_1 + a_2 + a_3 = 0}$</td>
<td>$(1,-T, 0)$</td>
</tr>
<tr>
<td>${(a_1,a_2T,a_3I)</td>
<td>a_1 \neq 0, a_1 + a_3 \neq 0, a_1 + a_2 + a_3 \neq 0}$</td>
<td>$(1,0,0)$</td>
</tr>
</tbody>
</table>

Theorem 14. For algebra system $\langle NT, \ast \rangle$, let $V_c = \{a| a \in NT \land \text{neut}(a) = c\}$, $V_{c \oplus d} = \{a \ast b| a, b \in NT \land \text{neut}(a) = c, \text{neut}(b) = d\}$, we have:

1. $V_c$ is a neutrosophic triplet subgroup of $NT$.
2. $V_{c \oplus d}$ is a neutrosophic triplet subgroup of $NT$.

Theorem 15. For an algebra system $\langle NT, \ast \rangle$, let $V_c = \{a| a \in NT \land \text{neut}(a) = c\}$, we have:

1. $V_c \cap V_d = \emptyset$ if $c \neq d$.
2. $NT = \bigcup_{c \in NS} V_c$. So $\bigcup_{c \in NS} V_c$ is a partition of $NT$, where $NS$ is a set, which contains all the neutral elements of $\langle NT, \ast \rangle$.

5. Conclusions

In the paper, we prove that $\langle NQ, \ast \rangle$ (or $\langle NQ, \ast \rangle$) is a neutrosophic extended triplet group, and provide new examples of a neutrosophic extended triplet group. We also explore the algebra structure properties of neutrosophic quadruple numbers. Moreover, we discuss two kinds of degenerate systems of neutrosophic quadruple numbers. For neutrosophic quadruple numbers, the results in the paper can be extended to general fields. In the following, we will explore the relation of neutrosophic quadruple numbers and other algebra systems [24–26]. Moreover, on the one hand, we will discuss the neutrosophic quadruple numbers based on some particular ring which can form a neutrosophic
extended triplet group, on the other hand, we will introduce a new operation \( \circ \) in order to guarantee \((NQ, \ast, \circ)\) is a neutrosophic triplet ring.

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**References**


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