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Strong Convergence of a System of Generalized Mixed Equilibrium Problem, Split Variational Inclusion Problem and Fixed Point Problem in Banach Spaces

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Abstract: The purpose of this paper is to introduce a new algorithm to approximate a common solution for a system of generalized mixed equilibrium problems, split variational inclusion problems of a countable family of multivalued maximal monotone operators, and fixed-point problems of a countable family of left Bregman, strongly asymptotically non-expansive mappings in uniformly convex and uniformly smooth Banach spaces. A strong convergence theorem for the above problems are established. As an application, we solve a generalized mixed equilibrium problem, split Hammerstein integral equations, and a fixed-point problem, and provide a numerical example to support better findings of our result.

Keywords: split variational inclusion problem; generalized mixed equilibrium problem; fixed point problem; maximal monotone operator; left Bregman asymptotically nonexpansive mapping; uniformly convex and uniformly smooth Banach space

1. Introduction and Preliminaries

Let $E$ be a real normed space with dual $E^*$. A map $B : E \to E^*$ is called:

(i) monotone if, for each $x, y \in E$, $\langle \eta - v, x - y \rangle \geq 0$, $\forall \, \eta \in Bx, \, v \in By$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing,

(ii) $\epsilon$-inverse strongly monotone if there exists $\epsilon > 0$, such that $\langle Bx - By, x - y \rangle \geq \epsilon \|Bx - By\|^2$,

(iii) maximal monotone if $B$ is monotone and the graph of $B$ is not properly contained in the graph of any other monotone operator. We note that $B$ is maximal monotone if, and only if it is monotone, and $R(J + tB) = E^*$ for all $t > 0$, $J$ is the normalized duality map on $E$ and $R(J + tB)$ is the range of $(J + tB)$ (cf. [1]).
Let $H_1$ and $H_2$ be Hilbert spaces. For the maximal monotone operators $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$, Moudafi [2] introduced the following split monotone variational inclusion:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*),$$

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*),$$

where $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are given operators. In 2000, Moudafi [3] proposed the viscosity approximation method, which is formulated by considering the approximate well-posed problem and combining the non-expansive mapping $S$ with a contraction mapping $f$ on a non-empty, closed, and convex subset $C$ of $H_1$. That is, given an arbitrary $x_1 \in C$, a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n,$$

converges strongly to a point of $F(S)$, the set of fixed point of $S$, whenever $\{\alpha_n\} \subset (0, 1)$ such that $\alpha_n \to 0$ as $n \to \infty$.

In [4,5], the viscosity approximation method for split variational inclusion and the fixed point problem in a Hilbert space was presented as follows:

$$u_n = J_{B_1}^{\lambda}(x_n + \gamma_n A^*(J_{B_2}^{\lambda} - I)Ax_n);$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n(u_n), \forall n \geq 1, \quad (1)$$

where $B_1$ and $B_2$ are maximal monotone operators, $J_{B_1}^{\lambda}$ and $J_{B_2}^{\lambda}$ are resolvent mappings of $B_1$ and $B_2$, respectively, $f$ is the Meir Keeler function, $T$ a non-expansive mapping, and $A^*$ is the adjoint of $A$, $\gamma_n, \alpha_n \in (0, 1)$ and $\lambda > 0$.

The algorithm introduced by Schopfer et al. [6] involves computations in terms of Bregman distance in the setting of $p$-uniformly convex and uniformly smooth real Banach spaces. Their iterative algorithm given below converges weakly under some suitable conditions:

$$x_{n+1} = \Pi_C^{-1}(f x_n + \gamma A^* (P Q - I)Ax_n), \forall n \geq 0, \quad (2)$$

where $\Pi_C$ denotes the Bregman projection and $P_C$ denotes metric projection onto $C$. However, strong convergence is more useful than the weak convergence in some applications. Recently, strong convergence theorems for the split feasibility problem (SFP) have been established in the setting of $p$-uniformly convex and uniformly smooth real Banach spaces [7–10].

Suppose that

$$F(x, y) = f(x, y) + g(x, y)$$

where $f, g : C \times C \to \mathbb{R}$ are bifunctions on a closed and convex subset $C$ of a Banach space, which satisfy the following special properties $(A_1) - (A_4), (B_1) - (B_3)$ and $(C)$:
\begin{align}
\begin{cases}
(A_1) & f(x, y) = 0, \forall x \in C; \\
(A_2) & f \text{ is maximal monotone}; \\
(A_3) & \forall x, y, z \in C \text{ and } t \in [0, 1] \text{ we have } \limsup_{n \to 0^+} (f(tx + (1-t)y) \leq f(x, y)); \\
(A_4) & \forall x \in C, \text{the function } y \mapsto f(x, y)\text{is convex and weakly lower semi-continuous}; \\
(B_1) & g(x, x) = 0 \; \forall \; x \in C; \\
(B_2) & g \text{ is maximal monotone, and weakly upper semi-continuous in the first variable}; \\
(B_3) & g \text{ is convex in the second variable}; \\
(C) & \text{for fixed } \lambda > 0 \text{ and } x \in C, \text{there exists a bounded set } K \subset C \\
\text{and } a \in K \text{ such that } f(a, z) + g(z, a) + \frac{1}{\lambda}(a - z, z - x) < 0 \; \forall x \in C \setminus K.
\end{cases}
\end{align}

The well-known, generalized mixed equilibrium problem (GMEP) is to find an \( x \in C \), such that
\[
F(x, y) + (Bx, y - x) \geq 0 \; \forall \; y \in C,
\]
where \( B \) is nonlinear mapping.

In 2016, Payvand and Jahedi \cite{11} introduced a new iterative algorithm for finding a common element of the set of solutions of a system of generalized mixed equilibrium problems, the set of common fixed points of a finite family of pseudo contraction mappings, and the set of solutions of the variational inequality for inverse strongly monotone mapping in a real Hilbert space. Their sequence is defined as follows:
\begin{align}
&g_i(u_{n,i}, y) + (C_i u_{n,i} + S_{n,i} x_n, y - u_{n,i}) + \theta_i(y) - \theta_i(u_{n,i}) \\
&+ \frac{1}{\tau_i}(y - u_{n,i}, u_{n,i} - x_n) \geq 0 \; \forall y \in K, \forall j \in I, \\
&y_n = \alpha_n u_n + (1 - \alpha_n (1 - f)) P_k (\sum_{j=0}^{\infty} \delta_{n,j} u_{n,j} - \lambda_n A \sum_{j=0}^{\infty} \delta_{n,j} u_{n,j}), \\
x_{n+1} = \beta_n x_n + (1 + \beta_n) (\gamma_0 + \sum_{j=1}^{\infty} \gamma_j T_j) P_k (y_n - \lambda_n A y_n) n \geq 1,
\end{align}
where \( g_i \) are bifunctions, \( S_i \) are \( \epsilon \)-inverse strongly monotone mappings, \( C_i \) are monotone and Lipschtz continuous mappings, \( \theta_i \) are convex and lower semicontinuous functions, \( A \) is a \( \Phi \)-inverse strongly monotone mapping, and \( f \) is an \( t \)-contraction mapping and \( \alpha_n, \delta_{n,j}, \beta_n, \lambda_n, \gamma_0 \in (0, 1) \).

In this paper, inspired by the above cited works, we use a modified version of (1), (2) and (4) to approximate a solution of the problem proposed here. Both the iterative methods and the underlying space used here are improvements and extensions of those employed in \cite{2,6,7,9–11} and the references therein.

Let \( p, q \in (1, \infty) \) be conjugate exponents, that is, \( \frac{1}{p} + \frac{1}{q} = 1 \). For each \( p > 1 \), let \( g(t) = t^{p-1} \) be a gauge function where \( g : R^+ \to R^+ \) with \( g(0) = 0 \) and \( \lim_{t \to \infty} g(t) = \infty \). We define the generalized duality map \( J_p : E \to 2^{E^*} \) by
\[
J_{g(t)}(x) = \{ x^* \in E^*; \langle x, x^* \rangle = \| x \| \| x^* \| , \| x^* \| = g(\| x \|) = \| x \|^{p-1} \}.
\]

In the sequel, \( a \vee b \) denotes \( \max \{a, b\} \).

\textbf{Lemma 1} \cite{12}. \textit{In a smooth Banach space } \( E \), the Bregman distance \( \Delta_p \) of \( x \) to \( y \), with respect to the convex continuous function \( f : E \to R \), such that \( f(x) = \frac{1}{p} \| x \|_p \), is defined by
\[
\Delta_p(x, y) = \frac{1}{q} \| x \|_p^q - \langle J_p(x), y \rangle + \frac{1}{p} \| y \|_p^p,
\]
for all \( x, y \in E \) and \( p > 1 \).
A Banach space $E$ is said to be uniformly convex if, for $x, y \in E, 0 < \delta_E(e) \leq 1$, where \( \delta_E(e) = \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq e, \text{ where } 0 \leq e \leq 2\} \).

**Definition 1.** A Banach space $E$ is said to be uniformly smooth, if for $x, y \in E$, \( \lim_{r \to 0} \left( \frac{\rho_E(r)}{r}\right) = 0 \) where \( \rho_E(r) = \frac{1}{\epsilon} \sup \{ \|x + y\| + \|x - y\| - 2 : \|x\| = 1, \|y\| \leq r; 0 \leq r < \infty \text{ and } 0 \leq \rho_E(r) < \infty \} \). It is shown in [12] that:

1. $\rho_E$ is continuous, convex, and nondecreasing with $\rho_E(0) = 0$ and $\rho_E(r) \leq r$
2. The function $r \mapsto \frac{\rho_E(r)}{r}$ is nondecreasing and fulfills $\frac{\rho_E(r)}{r} > 0$ for all $r > 0$.

**Definition 2 ([13]).** Let $E$ be a smooth Banach space. Let $\Delta_p$ be the Bregman distance. A mapping $T : E \to E$ is said to be a strongly non-expansive left Bregman with respect to the non-empty fixed point set of $T$, $F(T)$, if $\Delta_p(T(x), v) \leq \Delta_p(x, v) \forall x \in E$ and $v \in F(T)$.

Furthermore, if $\{x_n\} \subset C$ is bounded and $\lim_{n \to \infty} (\Delta_p(x_n, v) - \Delta_p(Tx_n, v)) = 0$, then it follows that $\lim_{n \to \infty} \Delta_p(x_n, Tx_n) = 0$.

**Definition 3.** Let $E$ be a smooth Banach space. Let $\Delta_p$ be the Bregman distance. A mapping $T : E \to E$ is said to be a strongly asymptotically non-expansive left Bregman with $\{k_n\} \subset [1, \infty)$ if there exists non-negative real sequences $\{k_n\}$ with $\lim_{n \to \infty} k_n = 1$, such that $\Delta_p(T^n(x), T^n(v)) \leq k_n \Delta_p(x, v), \forall (x, v) \in E \times F(T)$.

**Lemma 2 ([14]).** Let $E$ be a real uniformly convex Banach space, $K$ a non-empty closed subset of $E$, and $T : K \to K$ an asymptotically non-expansive mapping. Then, $I - T$ is demi-closed at zero, if $\{x_n\} \subset K$ converges weakly to a point $p \in K$ and $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$, then $p = Tp$.

**Lemma 3 ([12]).** In a smooth Banach space $E$, let $x_n \in E$. Consider the following assertions:

1. $\lim_{n \to \infty} \|x_n - x\| = 0$
2. $\lim_{n \to \infty} \|x_n\| = \|x\|$ and $\lim_{n \to \infty} \langle J^p(x_n), x \rangle = \langle J^p(x), x \rangle$
3. $\lim_{n \to \infty} \Delta_p(x_n, x) = 0$.

The implication $(1) \implies (2) \implies (3)$ are valid. If $E$ is also uniformly convex, then the assertions are equivalent.

**Lemma 4.** Let $E$ be a smooth Banach space. Let $\Delta_p$ and $V_p$ be the mappings defined by $\Delta_p(x, y) = \frac{1}{2}\|x\|^p - \langle J^p(x), y \rangle + \frac{1}{2}\|y\|^p$ for all $(x, y) \in E \times E$ and $V_p(x, x) = \frac{1}{2}\|x\|^q - \langle x, x \rangle + \frac{1}{2}\|x\|^p$ for all $(x, x^*) \in E \times E^*$. Then, $\Delta_p(x, y) = V_p(x, y)$ for all $x, y \in E$.

**Lemma 5 ([12]).** Let $E$ be a reflexive, strictly convex, and smooth Banach space, and $J^p$ be a duality mapping of $E$. Then, for every closed and convex subset $C \subset E$ and $x \in E$, there exists a unique element $\Pi_C^p(x) \in C$, such that $\Delta_p(x, \Pi_C^p(x)) = \min_{y \in C} \Delta_p(x, y)$; here, $\Pi_C^p(x)$ denotes the Bregman projection of $x$ onto $C$, with respect to the function $f(x) = \frac{1}{p}\|x\|^p$. Moreover, $x_0 \in C$ is the Bregman projection of $x$ onto $C$ if $\langle J^p(x_0 - x), y - x_0 \rangle \geq 0$

or equivalently

$\Delta_p(x_0, y) \leq \Delta_p(x, y) - \Delta_p(x, x_0)$ for every $y \in C$.

**Lemma 6 ([15]).** In the case of a uniformly convex space, $E$, with the duality map $J^q$ of $E^*$, $\forall x^*, y^* \in E^*$ we have

$\|x^* - y^*\|^q \leq \|x^*\|^q - q(J^q(x^*), y^*) + \sigma_q(x^*, y^*)$, where
\[ \sigma_q(x^*, y^*) = qG_q \int_0^1 \left( \frac{\|x^* - ty^*\|}{t} \right) \rho_{E^*} \left( \frac{\|y\|}{2(\|x^* - ty^*\| + \|y\|)} \right) dt \]

and \( G_q = 8 \vee 64c_kq^{-1} \) with \( c, k_q > 0 \).

**Lemma 7** ([12]). Let \( E \) be a reflexive, strictly convex, and smooth Banach space. If we write \( \Delta_q^*(x, y) = \frac{1}{q} \|x^*\|^q - (f_q^*, x^* + y^*) + \frac{1}{q} \|y^*\|^q \) for all \((x^*, y^*) \in E^* \times E^*\) for the Bregman distance on the dual space \( E^* \) with respect to the function \( f_q^* (x^*) = \frac{1}{q} \|x^*\|^q \), then we have \( \Delta_p(x, y) = \Delta_q^*(x^*, y^*) \).

**Lemma 8** ([16]). Let \( \{a_n\} \) be a sequence of non-negative real numbers, such that \( a_{n+1} \leq (1 - \beta_n)a_n + \delta_n \), \( n \geq 0 \), where \( \{\beta_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence in \( R \), such that

1. \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty \);
2. \( \limsup_{n \to \infty} \frac{\delta_n}{\beta_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then, \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 9.** Let \( E \) be reflexive, smooth, and strictly convex Banach space. Then, for all \( x, y, z \in E \) and \( x^*, z^* \in E^* \) the following facts hold:

1. \( \Delta_p(x, y) \geq 0 \) and \( \Delta_p(x, y) = 0 \) iff \( x = y \);
2. \( \Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle x^* - z^*, z - y \rangle \).

**Lemma 10** ([17]). Let \( E \) be a real uniformly convex Banach space. For arbitrary \( r > 1, \) let \( B_r(0) = \{ x \in E : \|x\| \leq r \} \). Then, there exists a continuous strictly increasing convex function

\[ g : [0, \infty) \to [0, \infty), g(0) = 0 \]

such that for every \( x, y \in B_r(0), f_x \in J_p(x), f_y \in J_p(y) \) and \( \lambda \in [0, 1], \) the following inequalities hold:

\[ \|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - (\lambda^p(1 - \lambda) + (1 - \lambda)^p\lambda)g(\|x - y\|) \]

and

\[ \langle x - y, f_x - f_y \rangle \geq g(\|x - y\|). \]

**Lemma 11** ([18]). Suppose that \( \sum_{n=1}^{\infty} \sup \{\|T_nz - T_ny\| : z \in C\} < \infty \). Then, for each \( y \in C, \{T_ny\} \) converges strongly to some point of \( C \). Moreover, let \( T \) be a mapping of \( C \) onto itself, defined by \( Ty = \lim_{n \to \infty} T_ny \) for all \( y \in C \). Then, \( \limsup_{n \to \infty} \{\|Tz - T_nz\| : z \in C\} = 0 \). Consequently, by Lemma 3, \( \limsup_{n \to \infty} \{\Delta_p(Tz, T_nz) : z \in C\} = 0 \).

**Lemma 12** ([19]). Let \( E \) be a reflexive, strictly convex, and smooth Banach space, and \( C \) be a non-empty, closed convex subset of \( E \). If \( f, g : C \times C \to \mathbb{R} \) be two bifunctions which satisfy the conditions \((A_1) - (A_4), (B_1) - (B_3) \) and \( (C) \), in (3), then for every \( x \in E \) and \( r > 0 \), there exists a unique point \( z \in C \) such that \( f(z, y) + g(z, y) + \frac{1}{r}(y - z, jz - jx) \geq 0 \) \( \forall \ y \in C \).

For \( f(x) = \frac{1}{r}\|x\|^p \), Reich and Sabach [20] obtained the following technical result:

**Lemma 13.** Let \( E \) be a reflexive, strictly convex, and smooth Banach space, and \( C \) be a non-empty, closed, and convex subset of \( E \). Let \( f, g : C \times C \to \mathbb{R} \) be two bifunctions which satisfy the conditions \((A_1) - (A_4), (B_1) - (B_3) \) and \( (C) \), in (3), then for every \( x \in E \) and \( r > 0 \), there exists a unique point \( z \in C \) such that

\[ f(z, y) + g(z, y) + \frac{1}{r}(y - z, jz - jx) \geq 0 \] \( \forall \ y \in C \).
\((A_4), (B_1) - (B_3)\) and \((C)\), in \((3)\). Then, for every \(x \in E\) and \(r > 0\), we define a mapping \(S_r : E \rightarrow C\) as follows;

\[
S_r(x) = \{z \in C : f(z, y) + g(z, y) + \frac{1}{r} \langle y - z, f_E^p(x - f_E^p) \rangle \geq 0, \forall y \in C\}. \tag{6}
\]

Then, the following conditions hold:

1. \(S_r\) is single-valued;
2. \(S_r\) is a Bregman firmly non-expansive-type mapping, that is,

\[
\forall x, y \in E \langle S_r x - S_r y, f_E^p(x - f_E^p) + f_E^p(y - f_E^p) \rangle \leq \langle S_r x - S_r y, f_E^p(x - f_E^p) \rangle
\]

or equivalently

\[
\triangle_p(S_r x, S_r y) + \triangle_p(S_r y, S_r x) + \triangle_p(S_r x, x) \leq \triangle_p(S_r x, y) + \triangle_p(S_r y, x);
\]
3. \(F(S_r) = \text{MEP}(f, g)\), here \(\text{MEP}\) stands for mixed equilibrium problem;
4. \(\text{MEP}(f, g)\) is closed and convex;
5. for all \(x \in E\) and for all \(v \in F(S_r)\), \(\triangle_p(v, S_r x) + \triangle_p(S_r x, x) \leq \triangle_p(v, x)\).

2. Main Results

Let \(E_1\) and \(E_2\) be uniformly convex and uniformly smooth Banach spaces and \(E_1^*\) and \(E_2^*\) be their duals, respectively. For \(i \in I\), let \(U_i : E_1 \rightarrow 2^{E_1^*}\) and \(T_i : E_2 \rightarrow 2^{E_2^*}\), \(i \in I\) be multi-valued maximal monotone operators. For \(i \in I\), \(\delta, \theta \in (1, \infty)\) and \(K \subseteq E_1\) closed and convex, let \(\Phi_i : K \times K \rightarrow \mathbb{R}, i \in I\), be bifunctions satisfying \((A1) - (A4)\) in \((3)\), let \(B_{\delta_i}^T : E_1 \rightarrow E_1\) be resolvent operators defined by \(B_{\delta_i}^T = (f_{E_1}^\delta + \delta U_i)^{-1}\) and \(B_{\delta_i}^T : E_2 \rightarrow E_2\) be resolvent operators defined by \(B_{\delta_i}^T = (f_{E_2}^\delta + \delta T_i)^{-1}\). Let \(A : E_1 \rightarrow E_2\) be a bounded and linear operator, \(A^*\) denotes the adjoint of \(A\) and \(AK\) be closed and convex. For each \(i \in I\), let \(S_i : E_1 \rightarrow E_1\) be a uniformly continuous Bregman asymptotically non-expansive operator with the sequences \(\{k_{i,j}\} \subset [1, \infty)\) satisfying \(\lim_{n \rightarrow \infty} k_{i,j} = 1\).

Denote by \(Y : E_1^* \rightarrow E_1^*\) a firmly non-expansive mapping. Suppose that, for \(i \in I\), \(\theta_i : K \rightarrow R\) are convex and lower semicontinuous functions, \(G_i : K \rightarrow E_1\) are \(\epsilon\)- inverse strongly monotone mappings and \(C_i : K \rightarrow E_1\) are monotone and Lipschitz continuous mappings. Let \(f : E_1 \rightarrow E_1\) be a \(\zeta\)-contraction mapping, where \(\xi \in (0, 1)\). Suppose that \(\Pi_{AK}^p : E_2 \rightarrow AK\) is a generalized Bregman projection onto \(AK\). Let \(\Omega = \{x^* \in \cap_{i=1}^\infty \text{SOLVIP}(U_i) : Ax^* \in \cap_{i=1}^\infty \text{SOLVIP}(T_i)\}\) be the set of solution of the split variational inclusion problem, \(\omega = \{x^* \in \cap_{i=1}^\infty \text{GMEP}(G_i, C_i, \theta_i, g)\}\) be the solution set of a system of generalized mixed equilibrium problems, and \(\Xi = \{x^* \in \cap_{i=1}^\infty \text{F}(S_i)\}\) be the common fixed-point set of \(S_i\) for each \(i \in I\). Let the sequence \(\{x_n\}\) be defined as follows:

\[
\begin{align*}
\Phi_l(x_{n+1}) &+ \langle f_{E_1} G_n x_n, y - u_{n,l} \rangle + \frac{1}{\delta} \langle y - u_{n,l}, f_{E_1}^p x_n - f_{E_1}^p x_{n+1} \rangle \geq 0, \forall y \in K, \\
\forall l &
i, \\
x_{n+1} &- I_{E_1^*} \left(\sum_{i=0}^\infty s_{n,i} B_{\delta_i}^T \left( f_{E_1} x_n - \sum_{i=0}^\infty \beta_{n,i} \lambda_n \lambda_n A^* f_{E_1}^p \left( I - \Pi_{AK}^p B_{\delta_i}^T \right) A u_{n,i} \right) \right), \tag{7}
\end{align*}
\]

where \(\Phi_l(x, y) = g_l(x, y) + \langle f_{E_1} C_l x, y - x \rangle + \theta_i(x) - \theta_i(x)\).

We shall strictly employ the above terminology in the sequel.

**Lemma 14.** Suppose that \(\overline{\zeta}\) is the function \((5)\) in Lemma 6 for the characteristic inequality of the uniformly smooth dual \(E_1^*\). For the sequence \(\{x_n\} \subset E_1\) defined by \((7)\), let \(0 \neq x_n \in E_1, 0 \neq A, 0 \neq f_{E_1} G_n x_n \in E_1^*\) and \(0 \neq \sum_{i=0}^\infty \beta_{n,i} f_{E_1}^p \left( I - \Pi_{AK}^p B_{\delta_i}^T \right) A u_{n,i} \in E_2^*\), \(i \in I\). Let \(\lambda_{n,i} > 0\) and \(r_{n,i} > 0\), \(i \in I\) be defined by

\[
\lambda_{n,i} = \frac{1}{\|A\| \|\sum_{i=0}^\infty \beta_{n,i} f_{E_1}^p \left( I - \Pi_{AK}^p B_{\delta_i}^T \right) A u_{n,i}\|}, \quad \text{and} \tag{8}
\]

\[
r_{n,i} = \frac{1}{\|f_{E_1} G_n x_n\|}, \tag{9}
\]

respectively.
Then for \( \mu_{n,i} = \frac{1}{\|x_n\|^{p-r}} \),

\[
2^d G_q \|J_E^p x_n\|^p \rho_{E_1} (\mu_{n,i}) \geq \begin{cases} \frac{1}{q} \|J_E^p x_n, r_n, J_E^p G_{n,j} x_n \| \\ \frac{1}{q} \|J_E^p x_n, r_n, A^{p-1} \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* \sum_{i=0}^{\infty} \beta_{n,i} J_E^p (I - \Pi_{AK} B_{E_i}^T) A u_{n,i} \| \
\end{cases}
\]

(10)

where \( G_q \) is the constant defined in Lemma 6 and \( \rho_{E_1} \) is the modulus of smoothness of \( E_1 \).

**Proof.** By Lemma 12, (6) in Lemma 13 and (7), for each \( i \in I \), we have that \( u_{n,i} = J_E^p (Y_{n,i} (J_E^p x_n - r_n J_E^p G_{n,j} x_n)) \). By Lemma 6, we get

\[
\frac{1}{q} \|J_E^p x_n, r_n, J_E^p G_{n,j} x_n \| = G_q \int_0^1 \frac{\||J_E^p x_n - t r_n J_E^p G_{n,j} x_n \| \|J_E^p x_n\|^q}{t} dt,
\]

(11)

for every \( t \in [0, 1] \).

However, by (9) and Definition 1(2), we have

\[
\rho_{E_1} \left( \|J_E^p x_n - t r_n J_E^p G_{n,j} x_n \| \|J_E^p x_n\| \right) \leq \rho_{E_1} \left( \|x_n\|^{p-1} \right) = \rho_{E_1} (t \mu_{n,i}).
\]

(12)

Substituting (12) into (11), and using the nondecreasing of function \( \rho_{E_1} \), we have

\[
\frac{1}{q} \|J_E^p x_n, r_n, J_E^p G_{n,j} x_n \| \leq 2^d G_q \|x_n\|^p \rho_{E_1} (\mu_{n,i}).
\]

(13)

In addition, by Lemma 6, we have

\[
\frac{1}{q} \|J_E^p x_n, \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* \sum_{i=0}^{\infty} \beta_{n,i} J_E^p (I - \Pi_{AK} B_{E_i}^T) A u_{n,i} \|
\]

\[
= G_q \int_0^1 \frac{\| \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* \sum_{i=0}^{\infty} \beta_{n,i} J_E^p (I - \Pi_{AK} B_{E_i}^T) A u_{n,i} \| \|J_E^p x_n\|^q}{t} dt,
\]

(14)

for every \( t \in [0, 1] \).

However, by (8) and Definition 1(2), we have

\[
\rho_{E_1} \left( \| \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* \sum_{i=0}^{\infty} \beta_{n,i} J_E^p (I - \Pi_{AK} B_{E_i}^T) A u_{n,i} \| \right)
\]

\[
\leq \rho_{E_1} \left( \|x_n\|^{p-1} \right) = \rho_{E_1} (t \mu_{n,i}).
\]

(15)
Substituting (15) into (14), and using the nondecreasing of function $\rho_{E_1}$, we get

$$\frac{1}{q} \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^*f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}$$

$$\leq 2^{1/2} \|x_n\|^p \rho_{E_1}(\mu_{n,i}).$$

(16)

By (13) and (16), the result follows. □

**Lemma 15.** For the sequence \( \{x_n\} \subset E_1 \), defined by (7), \( i \in I \), let \( 0 \neq \sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i} \in E_2^* \), \( 0 \neq f_{E_1}G_{n,i}x_n \in E_1^* \), and \( \lambda_n > 0 \) and \( r_{n,i} > 0 \), \( i \in I \), be defined by

$$\lambda_n = \frac{1}{\|A\|} \frac{1}{\|\sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}\|}$$

and

$$r_{n,i} = \frac{1}{\|f_{E_1}G_{n,i}x_n\|},$$

where \( i, \gamma \in (0, 1) \) and \( \mu_{n,i} \) are chosen such that

$$\rho_{E_1}(\mu_{n,i}) = \frac{i}{2^{1/2} \|x_n\|^p \|f_{E_1}G_{n,i}x_n\|} \times \frac{\|\sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}\|^p}{\|x_n\|^p \|\sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}\|^p}.$$

(19)

Then, for all \( v \in \Gamma \), we get

$$\Delta_p(x_{n+1}, v) \leq \Delta_p(x_n, v)$$

$$= [1 - i] \times \left( \frac{\|\sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}\|^p}{\|A\| \|\sum_{i=0}^{\infty} \beta_{n,i}f_{E_2}(I - \Pi_{AK}B_{n,i}^T)Au_{n,i}\|} \right)$$

and

$$\Delta_p(x_n, v) \leq [1 - \gamma] \times \frac{\|f_{E_1}G_{n,i}x_n - v\|}{\|f_{E_1}G_{n,i}x_n\|},$$

(21)

(22)

**Proof.** By Lemmas 13, 4 and 6, for each \( i \in I \), we get that \( u_{n,i} = (f_{E_1}G_{n,i}x_n - r_{n,i}f_{E_1}G_{n,i}x_n) \), and hence it follows that

$$\Delta_p(u_{n,i}, v) \leq V_p(f_{E_1}x_n - r_{n,i}f_{E_1}G_{n,i}x_n, v)$$

$$= -\langle f_{E_1}x_n, v \rangle + r_{n,i} \langle f_{E_1}G_{n,i}x_n, v \rangle$$

$$+ \frac{1}{q} \|f_{E_1}x_n - r_{n,i}f_{E_1}G_{n,i}x_n\|^q + \frac{1}{p} \|v\|^p.$$
By Lemmas 6 and 14, we have
\[
\frac{1}{q} \| J_{E_i}^p x_n - r_{n,i} J_{E_i}^p G_{n,i} x_n \|^q \\
\leq \frac{1}{q} \| J_{E_i}^p x_n \|^q - r_{n,i} \langle J_{E_i}^p G_{n,i} x_n, x_n \rangle + 2^q G_q \| J_{E_i}^p x_n \|^q \rho_{E_i}(\mu_{n,i}). \tag{24}
\]
Substituting (24) into (23), we have, by Lemma 4
\[
\triangle_p(u_{n,i}, v) \leq \triangle_p(x_n, v) + 2^q G_q \| J_{E_i}^p x_n \|^q \rho_{E_i}(\mu_{n,i}) \\
- r_{n,i} \langle J_{E_i}^p G_{n,i} x_n, x_n - v \rangle.
\tag{25}
\]
Substituting (18) and (20) into (25), we have
\[
\triangle_p(u_{n,i}, v) \leq \triangle_p(x_n, v) + \gamma \langle J_{E_i}^p G_{n,i} x_n, x_n - v \rangle \frac{\| J_{E_i}^p G_{n,i} x_n \|}{\| J_{E_i}^p G_{n,i} x_n \|} - \frac{\langle J_{E_i}^p G_{n,i} x_n, x_n - v \rangle}{\| J_{E_i}^p G_{n,i} x_n \|}
\]
\[
= \triangle_p(x_n, v) - [1 - \gamma] \times \frac{\langle J_{E_i}^p G_{n,i} x_n, x_n - v \rangle}{\| J_{E_i}^p G_{n,i} x_n \|}.
\]
Thus, (22) holds.

Now, for each \( i \in I \), let \( v = B_i^T v \) and \( Av = B_i^T Av \). By Lemma 4, we have
\[
\triangle_p(y_n, v) \leq \frac{1}{q} \left\| J_{E_i}^p u_{n,j} - \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^* J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i} \right\|^q + \frac{1}{p} \| v \|^p
\]
\[
- \langle J_{E_i}^p u_{n,j}, v \rangle + \left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^* J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i}, v \right\rangle,
\tag{26}
\]
where,
\[
\left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^* J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i}, v \right\rangle
\]
\[
= - \left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i}, (Av - \sum_{i=0}^{\infty} \beta_{n,i} A u_{n,i}) - \sum_{i=0}^{\infty} \beta_{n,i} (\Pi_{\mathcal{A}_i} B_i^{T_i} - I) A u_{n,i} \right\rangle
\]
\[
- \left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i}, \sum_{i=0}^{\infty} \beta_{n,i} (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i} \right\rangle
\]
\[
+ \left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n J_{E_i}^p (I - \Pi_{\mathcal{A}_i} B_i^{T_i}) A u_{n,i}, u_{n,i} \right\rangle.
\]
As \( \mathcal{A}_i \) is closed and convex, by Lemma 5 and the variational inequality for the Bregman projection of zero onto \( \mathcal{A}_i - \sum_{i=0}^{\infty} \beta_{n,i} A u_{n,i} \), we arrive at
\[
\left\langle \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n J_{E_i}^p (\Pi_{\mathcal{A}_i} B_i^{T_i} - I) A u_{n,i}, (Av - \sum_{i=0}^{\infty} \beta_{n,i} A u_{n,i}) - \sum_{i=0}^{\infty} \beta_{n,i} (\Pi_{\mathcal{A}_i} B_i^{T_i} - I) A u_{n,i} \right\rangle \geq 0
\]
Theorem 1.

Let $g_i : K \times K \to R$, $i \in I$, be bifunctions satisfying (A1) – (A4) in (3). For $\delta > 0$ and $p, q \in (1, \infty)$, let $(I - \Pi_{Ak}B_{\delta_i}^T)$, $i \in I$, be demi-closed at zero. Let $x_1 \in E_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows:

$$
\begin{align*}
&g_i(u_{n,i}, y) + \langle f_i^{p_i} G_{n,i} u_{n,i} + f_i^{p_i} G_{n,i} x_{n,y} - u_{n,i} \rangle + \theta_i(y) - \theta_i(u_{n,i}) \\
&\quad + \frac{1}{\rho_i} (y - u_{n,i}) - f_i^{p_i} u_{n,i} - f_i^{p_i} x_n \geq 0 \forall y \in K, \forall i \in I, \\
y_n = f_i^{p_i} \left( \sum_{i=0}^{\infty} \beta_{n,i} f_i^{p_i} \left( I - \Pi_{Ak}B_{\delta_i}^T \right) A u_{n,i} \right), \\
x_{n+1} = f_i^{p_i} \left( \eta_{n,i} f_i^{p_i} f_i^{p_i} (f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} f_i^{p_i} (S_{n,i}(y_n)) \right),
\end{align*}
$$

where $r_{n,i} = \frac{1}{\|f_i^{p_i} G_{n,i} x_{n,y} - y\|}$, $\mu_{n,i} = \frac{1}{\|x_n\|}$, and $\gamma \in (0, 1)$ such that $\rho_{E_1}^n (\mu_{n,i}) = \frac{\gamma (\|f_i^{p_i} G_{n,i} x_{n,y} - y\|)}{2 \|f_i^{p_i} G_{n,i} x_{n,y} - y\|}$.

$$
\lambda_n = \begin{cases} 
\frac{1}{\|A\|} & \frac{1}{\|f_i^{p_i} G_{n,i} x_{n,y} - y\|} \neq 0 \\
\frac{1}{\|A\|^{p_i}} & \frac{1}{\|f_i^{p_i} G_{n,i} x_{n,y} - y\|} = 0,
\end{cases}
$$

\(i \in (0, 1)\) and $\tau_{n,i} = \frac{1}{\|u_{n,i}\|}$ are chosen such that

$$
\rho_{E_1}^n (\tau_{n,i}) = \frac{i}{2 \|A\|^{p_i}} \times \frac{\| \sum_{i=0}^{\infty} \beta_{n,i} f_i^{p_i} (I - \Pi_{Ak}B_{\delta_i}^T) A u_{n,i} \|^{p_i}}{\|A\|^{p_i}}.
$$
with, \( \lim_{n \to \infty} \eta_{n,0} = 0, \eta_{n,0} \leq \sum_{i=1}^{\infty} \eta_{n,i} \) for \( M \geq 0, \eta_{n-1,0} \leq \sum_{i=1}^{\infty} \eta_{n-1,i} \leq \sum_{i=1}^{\infty} \eta_{n-1,i} M < \infty, \sum_{i=1}^{\infty} \eta_{n,i} = \sum_{i=1}^{\infty} a_{n,i} = \sum_{i=1}^{\infty} b_{n,i} = 1 \) and \( k_n = \max \{ k_{n,i} \} \). If \( \Gamma \cap \Omega \cap \Theta \neq \emptyset \), then \( \{ x_n \} \) converges strongly to \( x^* \in \Gamma \), where \( \sum_{i=0}^{\infty} b_{n,i} \Pi_{\beta_{n,i}}^{p} B_{\delta_{n,i}}^{T}(x^*) = \sum_{i=0}^{\infty} \beta_{n,i} B_{\delta_{n,i}}^{T}(x^*) \), for each \( i \in I \).

**Proof.** For \( x, y \in K \) and \( i \in I \), let \( \Phi_i(x, y) = g_i(x, y) + (\| f_{E_1}^{p} \|_C x, y - x \) + \( \theta_i(y) - \theta_i(x) \)). Since \( g_i \) are bi-functions satisfying (A1) - (A4) in (3) and \( C \) are monotone and Lipschitz continuous mappings, and \( \theta_i \) are convex and lower semicontinuous functions, therefore \( \Phi_i(i \in I) \) satisfy the conditions (A1) - (A4) in (3), and hence the algorithm (29) can be written as follows:

\[
\begin{align*}
\Phi_i(u_{n,i}, y) + \| f_{E_1}^{p} \|_C x_n, y - u_{n,i} \rangle + \frac{1}{r_{n,i}} \langle y - u_{n,i}, f_{E_1}^{p} u_{n,i} - f_{E_1}^{p} x_n \rangle & \geq 0 \\
\forall y \in K, \forall i \in I,
\end{align*}
\]

(32)

We will divide the proof into four steps.

**Step One:** We show that \( \{ x_n \} \) is a bounded sequence.

Assume that \( \| \sum_{i=0}^{\infty} \beta_{n,i} \Pi_{\beta_{n,i}}^{p} B_{\delta_{n,i}}^{T}(x_n) \| = 0 \) and \( \| f_{E_1}^{p} G_{n,i} x_n \| = 0 \). Then, by (32), we have

\[
\Phi_i(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, f_{E_1}^{p} u_{n,i} - f_{E_1}^{p} x_n \rangle \geq 0 \forall y \in K, \forall i \in I.
\]

(33)

By (33) and Lemma 13, for each \( i \in I \), we have that \( u_{n,i} = \Pi_{\beta_{n,i}}^{p} (Y_{r_{n,i}}(f_{E_1}^{p} x_n)) \). By Lemma 4 and for \( v \in \Gamma \) and \( v = Y_{r_{n,i}} \), we have

\[
\Delta_p(u_{n,i}, v) = V_{p}(Y_{r_{n,i}}(f_{E_1}^{p} x_n), v) \leq V_{p}(f_{E_1}^{p} x_n, v) = \Delta_p(x_n, v).
\]

(34)

In addition, for each \( i \in I \), let \( v = B_{\delta_{n,i}}^{T} v \). By Lemma 4 and for \( v \in \Gamma \), we have

\[
\Delta_p(y_n, v) = V_{p}(\sum_{i=0}^{\infty} \beta_{n,i} B_{\delta_{n,i}}^{T} v) \leq \Delta_p(u_{n,i}, v).
\]

(35)

Now assume that \( \| \sum_{i=0}^{\infty} \beta_{n,i} \Pi_{\beta_{n,i}}^{p} B_{\delta_{n,i}}^{T}(x_n) \| \neq 0 \) and \( \| f_{E_1}^{p} G_{n,i} x_n \| \neq 0 \). Then by (32), we have that

\[
\Phi_i(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, f_{E_1}^{p} u_{n,i} - (f_{E_1}^{p} x_n - r_{n,i} f_{E_1}^{p} G_{n,i} x_n) \rangle \geq 0 \forall y \in K, \forall i \in I.
\]

(36)

By (36) and Lemma 13, for each \( i \in I \), we have \( u_{n,i} = \Pi_{\beta_{n,i}}^{p} (Y_{r_{n,i}}(f_{E_1}^{p} x_n - r_{n,i} f_{E_1}^{p} G_{n,i} x_n)) \). For \( v \in \Gamma \), by (22) in Lemma 15, we get

\[
\Delta_p(u_{n,i}, v) \leq \Delta_p(x_n, v).
\]

(37)

In addition, for each \( i \in I \), \( v \in \Gamma \), (21) in Lemma 15 gives

\[
\Delta_p(y_n, v) \leq \Delta_p(u_{n,i}, v).
\]

(38)
Let \( u_{n,i} = 0 \). By Lemma 1, we have
\[
\triangle_p(u_{n,i}, v) = \frac{1}{p} \|v\|^p \tag{39}
\]
and by (27), (39), Lemmas 4 and 15, we have
\[
\triangle_p(y_n, v) \leq \frac{1}{q} \left\| \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^* \left( I - \Pi_{A^T_{B_0^T}}^p \right) A u_{n,i} \right\|^p
\]
\[
+ \triangle_p(u_{n,i}, v) + \lambda_n \left( \sum_{i=0}^{\infty} \beta_{n,i} \| E_2 (I - \Pi_{A^T_{B_0^T}}^p) A u_{n,i} \right.
\]
\[- \lambda_n \left( \sum_{i=0}^{\infty} \beta_{n,i} \| E_2 (I - \Pi_{A^T_{B_0^T}}^p) A u_{n,i} \right) \] \( (40) \)
However, by (30) and (40), we have
\[
\triangle_p(y_n, v)
\]
\[
\leq \frac{1}{q} \left\| A \right\|^p \left( \sum_{i=0}^{\infty} \beta_{n,i} \| E_2 (I - \Pi_{A^T_{B_0^T}}^p) A u_{n,i} \right)
\]
\[
+ \triangle_p(u_{n,i}, v) + \lambda_n \left( \sum_{i=0}^{\infty} \beta_{n,i} \| E_2 (I - \Pi_{A^T_{B_0^T}}^p) A u_{n,i} \right.
\]
\[- \lambda_n \left( \sum_{i=0}^{\infty} \beta_{n,i} \| E_2 (I - \Pi_{A^T_{B_0^T}}^p) A u_{n,i} \right) \] \( (41) \)
This implies that
\[
\triangle_p(y_n, v) \leq \triangle_p(u_{n,i}, v). \tag{42}
\]
By (42) and (37), we get
\[
\triangle_p(y_n, v) \leq \triangle_p(x_n, v). \tag{43}
\]
In addition, it follows from the assumption \( \eta_{n,0} \leq \sum_{i=1}^{\infty} \eta_{n,i} \), Definition 3, Lemmas 9 and 4
\[ \Delta_p(x_{n+1}, v) \]
\[ = \Delta_p \left( \int_{E_1}^p \left( \eta_{n,0} I_{E_1}^p(f(x_n)) + \sum_{i=1}^\infty \eta_{n,i} I_{E_1}^p(S_{n,i}(y_n)) \right), v \right) \]
\[ = V_p \left( \eta_{n,0} I_{E_1}^p(f(x_n)) + \sum_{i=1}^\infty \eta_{n,i} I_{E_1}^p(S_{n,i}(y_n)), v \right) \]
\[ \leq \eta_{n,0} V_p \left( I_{E_1}^p(f(x_n)), v \right) + \sum_{i=1}^\infty \eta_{n,i} V_p \left( I_{E_1}^p(S_{n,i}(y_n)), v \right) \)
\[ \leq \eta_{n,0} \Delta_p(x_n, v) + \sum_{i=1}^\infty \eta_{n,i} \Delta_p(y_n, v) \]
\[ + \left( \int_{E_1}^p x_n - I_{E_1}^p f(v), f(v) \right) \]
\[ \leq \eta_{n,0} \left( \Delta_p(f(v), v) + \left( I_{E_1}^p x_n - I_{E_1}^p f(v), f(v) \right) \right) \]
\[ + \left( \eta_{n,0} \xi + \sum_{i=1}^\infty \eta_{n,i} k_n \right) \Delta_p(x_n, v) \]
\[ \leq \eta_{n,0} \left( \Delta_p(f(v), v) + \left( I_{E_1}^p x_n - I_{E_1}^p f(v), f(v) \right) \right) \]
\[ + \left( \sum_{i=1}^\infty \eta_{n,i} (k_n + k_n) \right) \Delta_p(x_n, v) \]
\[ \leq \max \left\{ \frac{\Delta_p(f(v), v) + \left( I_{E_1}^p x_n - I_{E_1}^p f(v), f(v) \right)}{\xi + k_n}, \Delta_p(x_1, v) \right\}. \quad (44) \]

By (44), we conclude that \( \{ x_n \} \) is bounded, and hence, from (42), (34), (35), (44), (38), and (37), \( \{ y_n \} \) and \( \{ u_{n,i} \} \) are also bounded.

**Step Two:** We show that \( \lim_{m \to \infty} \Delta_p(x_{n+1}, x_n) = 0 \). By Lemmas 1, 4, 10, and 7, we have, by the convexity of \( \Delta_p \) in the first argument and for \( \eta_{n-1,0} \leq \sum_{i=1}^\infty \eta_{n-1,1,i} \)

\[ \Delta_p(x_{n+1}, x_n) = \Delta_p \left( \int_{E_1}^p \left( \eta_{n,0} I_{E_1}^p(f(x_n)) + \sum_{i=1}^\infty \eta_{n,i} I_{E_1}^p(S_{n,i}(y_n)) \right), v \right) \]
\[ \leq \eta_{n,0} \Delta_p \left( \int_{E_1}^p \left( f(x_n), v \right) \right) + \sum_{i=1}^\infty \eta_{n,i} \Delta_p \left( I_{E_1}^p \left( S_{n,i}(y_n), v \right) \right) \]
\[ + \sum_{i=1}^\infty \eta_{n,i} \left( \sum_{i=1}^{\infty} \eta_{n,i} \frac{1}{p} \| S_{n,i}(y_n) \|_p^p + \eta_{n,0} \| f(x_n) \|_p I_{E_1}^p \left( S_{n-1,i}(y_n), v \right) \right) \]
\[ + \eta_{n,0} \left( \sum_{i=1}^\infty \eta_{n,i} \frac{1}{p} \| f(x_n) \|_p + \sum_{i=1}^\infty \eta_{n,i} \| S_n(y_n) \|_p I_{E_1}^p \left( f(x_n), v \right) \right) \]
\[ + \sum_{i=1}^\infty \eta_{n,i} \left( \left( I_{E_1}^p S_{n,i}(y_n), I_{E_1}^p S_{n-1,i}(y_n) \right) \right) \]
\[ \leq (1 - \eta_{n,0} (1 - \xi)) \Delta_p(x_n, x_{n+1}) + \sum_{i=1}^\infty \eta_{n,i} \sup_{\eta_{n-1,i} \geq 1} \left\{ \Delta_p(S_{n,i}(y_n), S_{n-1,i}(y_n)) \right\} \]
\[ + \sum_{i=1}^\infty \eta_{n,i} M, \quad (45) \]
where
\[ M = \max \left\{ \max \{ \| f(x_n) \|, \| S_{n-1,i}(y_{n-1}) \| \}, \max \{ \| f(x_{n-1}) \|, \| S_{n,i}(y_n) \| \} \right\}. \]

In view of the assumption \( \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \eta_{n-1,i} M < \infty \) and (45), Lemmas 11 and 8 imply
\[ \lim_{n \to \infty} \Delta_p(x_{n+1}, x_n) = 0. \tag{46} \]

**Step Three:** We show that \( \lim_{n \to \infty} \Delta_p(S_n y_n, y_n) = 0. \)

For each \( i \in I \), we have
\[ \Delta_p(S_i(y_n), v) \leq \Delta_p(y_n, v). \]

Then,
\[
\begin{align*}
0 & \leq \Delta_p(y_n, v) - \Delta_p(S_i(y_n), v) \\
& = \Delta_p(y_n, v) - \Delta_p(x_{n+1}, v) + \Delta_p(x_{n+1}, v) - \Delta_p(S_i(y_n), v) \\
& \leq \Delta_p(x_n, v) - \Delta_p(x_{n+1}, v) + \Delta_p(x_{n+1}, v) - \Delta_p(S_i(y_n), v) \\
& = \Delta_p(x_n, v) - \Delta_p(x_{n+1}, v) + \Delta_p \left( J^p_{E1} \left( \eta_{n,0} J^p_{E1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E1}(S_{n,i}(y_n)) \right), v \right) \\
& \leq \Delta_p(x_n, v) - \Delta_p(x_{n+1}, v) + \eta_{n,0} \Delta_p(f(x_n), v) - \eta_{n,0} \Delta_p(S_i(y_n), v) \\
& \to 0 \text{ as } n \to \infty. \tag{47}
\end{align*}
\]

By (47) and Definition 2, we get
\[ \lim_{n \to \infty} \Delta_p(S_i y_n, y_n) = 0. \tag{48} \]

By uniform continuity of \( S \), we have
\[ \lim_{n \to \infty} \Delta_p(S_n y_n, y_n) = 0. \tag{49} \]

**Step Four:** We show that \( x_n \to x^* \in \Gamma. \)

Note that,
\[
\begin{align*}
\Delta_p(x_{n+1}, y_n) &= \Delta_p \left( J^p_{E1} \left( \eta_{n,0} J^p_{E1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E1}(S_{n,i}(y_n)) \right), y_n \right) \\
& \leq \eta_{n,0} \Delta_p(f(x_n), y_n) + \sum_{i=1}^{\infty} \eta_{n,i} \Delta_p(S_{n,i}(y_n), y_n) \\
& \leq \eta_{n,0} \zeta \Delta_p(x_n, y_n) + \Delta_p(f(y_n), y_n) + (f(x_n) - f(y_n), J^p_{E1} f(y_n) - J^p_{E1} y_n) \\
& + \sum_{i=1}^{\infty} \eta_{n,i} \Delta_p(S_{n,i}(y_n), y_n) \\
& \leq (1 - \eta_{n,0}(1 - \zeta)) \Delta_p(x_n, y_n) \\
& + \eta_{n,0}(\Delta_p(f(y_n), y_n) + (f(x_n) - f(y_n), J^p_{E1} f(y_n) - J^p_{E1} y_n)) \\
& + \sum_{i=1}^{\infty} \eta_{n,i} \Delta_p(S_{n,i}(y_n), y_n). \tag{50}
\end{align*}
\]
By (49), (50), and Lemma 8, we have
\[
\lim_{n \to \infty} \Delta_p(x_n, y_n) = 0. 
\] (51)

Therefore, by (51) and the boundedness of \( \{y_n\} \), and since by (46), \( \{x_n\} \) is Cauchy, we can assume without loss of generality that \( y_n \to x^* \) for some \( x^* \in E_1 \). It follows from Lemmas 2, 3, and (48) that \( x^* = S_i x^*, \) for each \( i \in I \). This means that \( x^* \in \Omega \).

In addition, by (31) and the fact that \( u_{n,i} \to x^* \) as \( n \to \infty \), we arrive at
\[
\frac{(J_{E_1}^p u_{n,i} - J_{E_1}^p y_n) - \sum_{i=0}^\infty \beta_{n,i} \lambda_n A^* B_{E_2}^T (I - \Pi_{Ak}^p B_{E_0}^T) A u_{n,i}}{\delta_n} \in \sum_{i=0}^\infty \alpha_{n,i} U_i(y_n). 
\] (52)

By (21), we have
\[
\| \sum_{i=0}^\infty \beta_{n,i} (I - \Pi_{Ak}^p B_{E_0}^T) A u_{n,i} \| \leq \left[ \frac{\Delta_p(u_{n,i}, v) - \Delta_p(y_n, v)}{\|A\|^{-1} |1 - t|} \right] \to 0 \text{ as } n \to \infty, 
\] (53)
and by (41), we have
\[
\| \sum_{i=0}^\infty \beta_{n,i} (I - \Pi_{Ak}^p B_{E_0}^T) A u_{n,i} \| \leq \left[ \frac{\Delta_p(u_{n,i}, v) - \Delta_p(y_n, v)}{(p \|A\|)^{-1}} \right] \to 0 \text{ as } n \to \infty. 
\] (54)

From (53), (54), and (52), by passing \( n \) to infinity in (52), we have that \( 0 \in \sum_{i=0}^\infty \sum_{i=0}^\infty \sum_{i=0}^\infty \sum_{i=0}^\infty \beta_{n,i}(I - \Pi_{Ak}^p B_{E_0}^T) \) at zero, we have that \( 0 \in \sum_{i=0}^\infty \beta_{n,i} T_i(Ax^*). \) Therefore, \( Ax \in SOLVIP(T_i) \) as \( \sum_{i=0}^\infty \beta_i = \sum_{i=0}^\infty \beta_{n,i} B_{E_0}^T (Ax^*) = \sum_{i=0}^\infty \beta_{n,i} B_{E_0}^T (Ax^*). \) This means that \( x^* \in \Omega. \)

Now, we show that \( x^* \in (\bigcap_{i=1}^\infty \text{GMEP}(\theta_i, C_i, G_i, \delta_i)). \) By (32), we have
\[
\Phi_i(u_{n,i}, y) + \langle J_{E_1}^p G_{n,i} x_n, y - u_{n,i} \rangle + \frac{1}{r_{n,i}} (y - u_{n,i}, J_{E_1}^p u_{n,i} - J_{E_1}^p x_n) \geq 0 
\forall y \in K, \forall i \in I, 
\]

Since \( \Phi_i \), for each \( i \in I \), are monotone, that is, for all \( y \in K, \)
\[
\Phi_i(u_{n,i}, y) + \Phi_i(y, u_{n,i}) \leq 0 
\]
\[
\Rightarrow \frac{1}{r_{n,i}} (y - u_{n,i}, J_{E_1}^p u_{n,i} - J_{E_1}^p x_n) \geq \Phi_i(y, u_{n,i}) + \langle J_{E_1}^p G_{n,i} x_n, y - u_{n,i} \rangle, 
\]
therefore,
\[
\frac{1}{r_{n,i}} (y - u_{n,i}, J_{E_1}^p u_{n,i} - J_{E_1}^p x_n) \geq \Phi_i(y, u_{n,i}) + \langle J_{E_1}^p G_{n,i} x_n, y - u_{n,i} \rangle. 
\]

By the lower semicontinuity of \( \Phi_i \), for each \( i \in I \), the weak upper semicontinuity of \( G \), and the facts that, for each \( i \in I \), \( u_{n,i} \to x^* \) as \( n \to \infty \) and \( J^p \) is norm – lo – weak* uniformly continuous on a bounded subset of \( E_1 \), we have
\[
0 \geq \Phi_i(y, x^*) + \langle J_{E_1}^p G_{n,i} x^*, y - x^* \rangle. 
\] (55)
Now, we set \( y = ty + (1 - t)x^* \in K \). From (55), we get
\[
0 \geq \Phi_i(y, x^*) + \langle f_{E_i}^p, G_{n,j}x^*, y - x^* \rangle.
\] (56)

From (56), and by the convexity of \( \Phi_i \), for each \( i \in I \), in the second variable, we arrive at
\[
0 = \Phi_i(y_i, y_i) \leq t \Phi_i(y_i, y) + (1 - t)\Phi_i(y_i, x^*)
\]
\[
\leq t \Phi_i(y_i, y) + (1 - t)\langle f_{E_i}^p, G_{n,j}x^*, y - x^* \rangle
\]
\[
\leq t \Phi_i(y_i, y) + (1 - t)\langle f_{E_i}^p, G_{n,j}^*x^*, y - x^* \rangle,
\]
which implies that
\[
\Phi_i(y_i, y) + (1 - t)\langle f_{E_i}^p, G_{n,j}^*x^*, y - x^* \rangle \geq 0.
\] (57)

From (57), by the lower semicontinuity of \( \Phi_i \), for each \( i \in I \), we have for \( y_i \to x^* \) as \( t \to 0 \)
\[
\Phi_i(x^*, y) + \langle f_{E_i}^p, G_{n,j}x^*, y - x^* \rangle \geq 0.
\] (58)

Therefore, by (58) we can conclude that \( x^* \in (\cap_{i=1}^\infty \text{GMEP}(\theta_i, C_i, G_i, g_i)). \) This means that \( x^* \in \omega \). Hence, \( x^* \in \Gamma \).

Finally, we show that \( x_n \to x^* \), as \( n \to \infty \). By Definition 3, we have
\[
\Delta_p(x_{n+1}, x^*)
\]
\[
= \Delta_p(f_{E_i}^p) \left( \eta_{n,0} f_{E_i}^p(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} f_{E_i}^p(G_{n,j}(y_n)) \right), x^*)
\]
\[
\leq \eta_{n,0} \Delta_p^q(f_{E_i}^p(f(u_n)), f_{E_i}^p(x^*)) + \sum_{i=1}^{\infty} \eta_{n,i} \Delta_p^q(f_{E_i}^p(G_{n,j}(y_n)), f_{E_i}^p(x^*)
\]
\[
\leq \eta_{n,0} \Delta_p(x_n, x^*) + \eta_{n,0} \Delta_p(f(x^*, x^*) + \sum_{i=1}^{\infty} \eta_{n,i} k_n \Delta_p(y_n, x^*)
\]
\[
+ \langle f_{E_i}^p(x_n) - f_{E_i}^p, f(x^*) - x^* \rangle + \sum_{i=1}^{\infty} \eta_{n,i} k_n \Delta_p(y_n, x^*)
\]
\[
\leq \eta_{n,0} \Delta_p(f(x^*, x^*) + \langle f_{E_i}^p(x_n) - f_{E_i}^p, f(x^*) - x^* \rangle
\]
\[
+ \left(1 - \sum_{i=1}^{\infty} \eta_{n,i} (1 - k_n) \right) \Delta_p(x_n, x^*).
\] (59)

By (59) and Lemma 8, we have that
\[
\lim_{n \to \infty} \Delta_p(x_n, x^*) = 0.
\]

The proof is completed. \( \square \)

In Theorem 1, \( i = 0 \) leads to the following new result.

**Corollary 1.** Let \( g : K \times K \to R \) be bifunctions satisfying (A1) – (A4) in (3). Let \( (1 - \Pi_{AK}^p B_{\delta}^T) \) be demiclosed at zero. Suppose that \( x_1 \in E_1 \) is chosen arbitrarily and the sequence \( \{x_n\} \) is defined as follows:
\[
\begin{align*}
&\varrho(u_n, y) + \langle f_{E_i}^p, C_{u_n} + f_{E_i}^p, G_{n,xn}, y - u_n \rangle + \theta(y) = \theta(u_n) \\
&+ \frac{1}{t_n}(y - u_n) f_{E_i}^p, u_n - f_{E_i}^p, x_n \rangle \geq 0 \forall y \in K, \\
&y_n = f_{E_i}^p \left( B_{\delta}^T \left( f_{E_i}^p, u_n - \lambda_n A^T \right) I_{E_i}^p (1 - \Pi_{AK}^p B_{\delta}^T), \right) \\
x_{n+1} = f_{E_i}^p \left( \eta_n f_{E_i}^p(f(x_n)) + (1 - \eta_n) f_{E_i}^p(S_n(y_n)) \right) n \geq 1,
\end{align*}
\] (60)
Then, the mapping $D$ is maximal monotone.

Let $E$ be a Banach space. Let $F, K : E \to E$ be bounded and maximal monotone operators. Let $D : E \times E^* \to E^* \times E$ be defined by $D(x, y) = (Fx - y, Ky + x)$ for all $(x, y) \in E \times E^*$. Then, the mapping $D$ is maximal monotone.

### Lemma 16 ([21])

Let $E$ be a Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be bounded and maximal monotone operators. Let $D : E \times E^* \to E^* \times E$ be defined by $D(x, y) = (Fx - y, Ky + x)$ for all $(x, y) \in E \times E^*$. Then, the mapping $D$ is maximal monotone.
By Lemma 16, if \( K, K', \) and \( F, F' \) are multi-valued maximal monotone operators then, we have two resolvent mappings,

\[
B^D_\delta = (J^p_{E_1} + \delta J^p_{E_1})^{-1} J^p_{E_1}, \quad \text{and} \quad B^{D'}_\delta = (J^p_{E_2} + \delta J^p_{E_2})^{-1} J^p_{E_2},
\]

where \( F : E_1 \rightarrow E_1^* \), \( K : E_1^* \rightarrow E_1 \) are multi-valued and maximal monotone operators, \( D : E_1 \times E_1^* \rightarrow E_1 \) is defined by \( D(x,y) = (Fx - y, Ky + x) \) for all \( (x,y) \in E_1 \times E_1^* \), and \( F' : E_2 \rightarrow E_2^* \), \( K' : E_2^* \rightarrow E_2 \) are multi-valued and maximal monotone operators, \( D' : E_2 \times E_2^* \rightarrow E_2 \times E_2 \) is defined by \( D'(Ax,Ay) = (F'Ax - Ay, K'Ay + Ax) \) for all \( (Ax,Ay) \in E_2 \times E_2^* \). Then \( D \) and \( D' \) are maximal monotone by Lemma 16.

When \( U = D \) and \( T = D \) in Corollary 1, the algorithm (60) becomes

\[
\begin{align*}
&\left\{ g(u_n, y) + \langle J^p_{E_1} C_n u_n + J^p_{E_1} G_n x_n, y - u_n \rangle + \theta(y) - \theta(u_n) \\
&\quad + \frac{1}{r_n} \langle y - u_n, J^p_{E_1} u_n - J^p_{E_1} x_n \rangle \geq 0 \quad \forall y \in K,
\end{align*}
\]

\[
\begin{align*}
&y_n = J^p_{E_1} \left( \frac{b_{S_n}}{p} \left( J^p_{E_1} u_n - \lambda_n A^* J^p_{E_2} (I - \Pi_{A_{n+1}}^p B^D_{S_n}) A u_n \right) \right), \\
x_{n+1} = J^p_{E_1} \left( \eta_n J^p_{E_1} (f(x_n)) + (1 - \eta_n) J^p_{E_1} (S_n(y_n)) \right) \\
\end{align*}
\]

and its strong convergence is guaranteed, which solves the problem of a common solution of a system of generalized mixed equilibrium problems, split Hammerstein integral equations, and fixed-point problems for the mappings involved in this algorithm.

4. A Numerical Example

Let \( i = 0, E_1 = E_2 = \mathbb{R}, \) and \( K = AK = [0, \infty), \) for \( Ax = x \ \forall x \in E_1. \) The generalized mixed equilibrium problem is formulated as finding a point \( x \in K \) such that,

\[
g_0(x, y) + \langle G_0 y - x \rangle + \theta_0(y) - \theta_0(x) \geq 0, \quad \forall y \in K. \quad (64)
\]

Let \( r_0 \in (0, 1] \) and define \( \theta_0 = 0, g_0(x, y) = \frac{y^2}{r_0} + \frac{2x^2}{r_0} \) and \( G_0(x) = S_0(x) = \frac{1}{r_0} x. \)

Clearly, \( g_0(x, y) \) satisfies the conditions \((A1) - (A4)\) and \( G_0(x) = S_0(x) \) is a Bregman asymptotically non-expansive mapping, as well as a \( 1 \)- inverse strongly monotone mapping. Since \( Y_{r_0} \) is single-valued, therefore for \( y \in K, \) we have that

\[
g_0(u_0, y) + \langle G_0 y - u_0 \rangle + \frac{1}{r_0} (y - u_0, u_0 - x) \geq 0
\]

\[
\iff \frac{y^2}{r_0} + \frac{2u_0^2}{r_0} + \frac{1}{r_0} (y - u_0, u_0) \geq 0 \\
\iff \frac{y^2}{r_0} + \frac{2|u_0|^2}{r_0^2} + \frac{x^2}{r_0} \geq 0. \quad (65)
\]

As (65) is a nonnegative quadratic function with respect to \( y \) variable, so it implies that the coefficient of \( y^2 \) is positive and the discriminant \( \frac{4u_0^2}{r_0^2} - \frac{4x^2}{r_0^2} \leq 0, \) and therefore \( u_0 = x \sqrt{r_0}. \) Hence,

\[
Y_{r_0}(x) = x \sqrt{r_0}. \quad (66)
\]
By Lemma 13 and (66), $F(Y_{r_0}) = GEP(g_0, G_0) = \{0\}$ and $F(S_0) = \{0\}$. Define

$$U_0, T_0 : \mathbb{R} \rightarrow \mathbb{R} \text{ by } U_0(x) = T_0(Ax) \begin{cases} (0, 1), x \geq 0 \\ (1), x < 0, \end{cases}$$

$$P_{[0, \infty)} : \mathbb{R} \rightarrow [0, \infty) \text{ by } P_{[0, \infty)}(Ax) = \begin{cases} 0, Ax \in (-\infty, 0) \\ Ax, Ax \in [0, \infty), \end{cases}$$

$$B^T_0 : \mathbb{R} \rightarrow \mathbb{R} \text{ by } B^T_0(y) = \frac{\frac{y}{1+y}}{1+y}, y \geq 0,$$

$$P_{[0, \infty)}B^T_0 : \mathbb{R} \rightarrow [0, \infty) \text{ by } P_{[0, \infty)}B^T_0(y) = \begin{cases} \frac{Ay}{1+y}, Ay \geq 0 \\ 0, Ay < 0. \end{cases}$$

It is clear that $U_0$ and $T_0$ are multi-valued maximal monotone mappings, such that $0 \in SOLVIP(U_0)$ and $0 \in SOLVIP(T_0)$. We define the $\zeta$-contraction mapping by $f(x) = \frac{x}{2}$, $\delta_n = \frac{1}{2^{n+1}}$, $\eta_{n,0} = \frac{1}{n+1}$, $r_{n,0} = \frac{1}{2n}$ and $\zeta = \frac{1}{2}$. Hence, for

$$\lambda_n = \begin{cases} 1 + \left(0, \frac{1}{2^{n+1}}\right) - u_{n,0}, & u_{n,0} > 0, \\ 1, u_{n,0} = 0, \\ \frac{1}{|u_{n,0}|}, u_{n,0} < 0, \end{cases}$$

$$\begin{cases} u_{n,0} = \frac{1}{2} x_n, \\ y_n = \frac{u_{n,0}}{1 + \left(0, \frac{1}{2^{n+1}}\right)} (u_{n,0} - 1), u_{n,0} > 0, \\ y_n^2 = \left[\frac{u_{n,0}}{1 + \left(0, \frac{1}{2^{n+1}}\right)}\right]^2, u_{n,0} = 0, \\ y_n^3 = \frac{2u_{n,0} + 1}{2^{n+1}}, u_{n,0} > 0, \\ x_{n+1} = \frac{x_n}{2(n+1)} + \frac{2a_{n,y_n}}{(n+1)}, n \geq 1, \end{cases}$$

we get,

$$x_{n+1} = \begin{cases} \frac{x_n}{2(n+1)} + \frac{a_{n,y_n}^2 - 2a_{n,y_n}}{(n+1)\left(1 + \left(0, \frac{1}{2^{n+1}}\right)\right)}, x_n > 0, \\ \frac{x_n}{2(n+1)} + \frac{a_{n,y_n}^2}{(n+1)\left(1 + \left(0, \frac{1}{2^{n+1}}\right)\right)}, x_n = 0, \\ \frac{x_n}{2(n+1)} + \frac{n^{2n+1}a_{n,y_n}}{(2n+1)^2}, x_n < 0. \end{cases}$$

In particular,

$$x_{n+1} = \begin{cases} \frac{x_n}{2(n+1)} + \frac{5(n^{2n+1}a_{n,y_n})}{6(n+1)}, x_n > 0, \\ \frac{x_n}{2(n+1)} + \frac{5n^{2n+1}a_{n,y_n}}{6(n+1)}, x_n = 0, \\ \frac{x_n}{2(n+1)} + \frac{n^{2n+1}a_{n,y_n}^2}{2^{n+1}}, x_n < 0. \end{cases}$$

By Theorem 1, the sequence $\{x_n\}$ converges strongly to $0 \in \Gamma$. The Figures 1 and 2 below obtained by (MATLAB) software indicate convergence of $\{x_n\}$ given by (32) with $x_1 = -10.0$ and $x_1 = 10.0$, respectively.
Figure 1. Sequence convergence with initial condition $-10.0$.

Figure 2. Sequence convergence with initial condition $10.0$

Remark 1. Our results generalize and complement the corresponding ones in [2,7,9,10,22,23].

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