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The Gravitational Magnetoelectric Effect

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Abstract: Electromagnetism in spacetime can be treated in terms of an analogue linear dielectric medium. In this paper, we discuss the gravitational analogue of the linear magnetoelectric effect, which can be found in multiferroic materials. While this is known to occur for metrics with non-zero mixed components, we show how it depends on the choice of spatial formalism for the electromagnetic fields, including differences in tensor weight, and also on the choice of coordinate chart. This is illustrated for Langevin–Minkowski, four charts of Schwarzschild spacetime, and two charts of pp gravitational waves.

Keywords: magnetoelectric effect; general relativity; analogue models

1. Introduction

In a linear dielectric medium, polarisation and magnetisation depend linearly on electric and magnetic fields, respectively. However, it is also possible for a magnetic field to induce polarisation, and for an electric field to induce magnetisation. In the following form, this is known as the linear magnetoelectric effect (cf. Landau and Lifshitz [1]),

$$P^i = \varepsilon_0 \chi_e^{ij} E_j + \alpha^{ij} H_j \,, \tag{1}$$

$$\mu_0 M^i = \mu_0 \chi_m^{ij} H_j + \alpha^{ji} E_j \,, \tag{2}$$

where the standard electric and magnetic susceptibilities are denoted by χ_e^{ij} and χ_m^{ij}, respectively, and the magnetoelectric effect is described by α^{ij}. The first example of a material with an intrinsic magnetoelectric effect, Cr_2O_3, was found by Dzyaloshinskii [2] and Astrov [3]. More recently, multiferroics such as $GaFeO_3$ were found to exhibit a much stronger magnetoelectric effect. In particular, Sawada and Nagaosa [4] showed that this gives rise to a Lorentz-type force acting on light, which yields an optical magnetoelectric effect that can produce polarisation-independent birefringence of light. For a review of the magnetoelectric material science and energy conditions, see, e.g., [5].

In this article, we discuss the analogue of the linear magnetoelectric effect for electromagnetism in curved spacetimes. It is well-known that such an effect occurs for metrics with non-zero mixed time-space components g_{0i} (e.g., [6,7]), and that this corresponds to a magnetoelectric or moving medium (see, e.g., [8] for a recent review, and [9] for the metric approach to transformation optics). Resulting optical effects, such as rotation of the plane of polarisation for rotating spacetimes, have been studied already in the early literature (e.g., [6,10]), even before the Kerr solution was found (cf. [7]; for a more recent discussion of the gravitational Faraday effect see, e.g., [11,12]). However, since the electric and magnetic susceptibilities and the magnetoelectric effect in Equations (1) and (2) are spatial, they depend on the definition of the spatial electromagnetic fields. However, this definition can be done in several ways, resulting in a subtle difference between tensor fields and tensor density fields.

Thus, the main purpose of the present article is to clarify this dependence by explicitly computing and comparing the gravitational magnetoelectric effects α^{ij} for different choices of spatial electromagnetic fields, and coordinate charts, which can, of course, capture a moving medium as well. We begin by reviewing two choices of spatial formalism in Section 2, followed by the identification of the corresponding gravitational magnetoelectric effects, in addition to the relative permittivities and permeabilities, in Section 3. This shows that, irrespective of the formalism considered, the relative permittivities equal the relative permeabilities.

Moreover, while the gravitational magnetoelectric effect is well-known for rotating spacetimes such as the Kerr, as mentioned above, it is perhaps surprising that it also occurs for suitable charts of the static Schwarzschild, and even Minkowski spacetime. Thus, we discuss the implications for the rotating Langevin form of Minkowski, four coordinate charts of the Schwarzschild spacetime, and two charts of pp gravitational waves in Section 4, to exhibit explicitly the dependence of these quantities on the choice of spatial formalism and of chart. Finally, we conclude in Section 5.

Throughout the paper, we use Greek indices for spacetime and Latin indices for space, apply Einstein's convention for summing over repeated indices, and employ the metric signature $(-,+,+,+)$. We also use Levi–Civita symbols in three-space, as totally antisymmetric tensor densities with $\epsilon^{123} = 1$, $\epsilon_{123} = 1$ as usual. Regarding units, we set Rømer's constant (the speed of light) $c = (\varepsilon_0\mu_0)^{-\frac{1}{2}} = 1$. With this choice, it may be noted that the components of the magnetoelectric effect α^{ij}, having dimension TL^{-1} in SI units, are dimensionless similar to the susceptibilities. Moreover, $[E] = [B]$ and $[D] = [H]$.

2. Spacetime as a Medium

2.1. Constitutive Tensor Density

Electromagnetism in a linear medium can be described by the field tensor $F_{\mu\nu}$ and

$$G^{\alpha\beta} = \frac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}\,, \tag{3}$$

where $\chi^{\alpha\beta\gamma\delta}$ is called the constitutive tensor density (e.g., Post [13], Chapter 6), which characterises the properties of the medium and has area metric symmetries

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta}\,, \quad \chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta}\,, \quad \chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma}\,. \tag{4}$$

Constitutive relations of the form in Equation (3) have a long history, occurring already in Bateman's discussion of Kummer's quartic surface [14], and are the subject of premetric electrodynamics (e.g., [15]). For a discussion of area metric electromagnetism and more general tensorial backgrounds, see, e.g., [16]. Note also that the symmetries in Equation (4) imply that

$$G^{\mu\nu} = 2\frac{\delta}{\delta F_{\mu\nu}}\int \mathrm{d}^4x\,\mathcal{L} \quad \text{where} \quad \mathcal{L} = \frac{1}{8}\chi^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,. \tag{5}$$

Then Maxwell's equations in the absence of charges and currents are

$$\partial_{[\alpha}F_{\beta\gamma]} = 0\,, \tag{6}$$

$$\partial_\beta G^{\alpha\beta} = 0\,. \tag{7}$$

Now, if the medium is simply a vacuum spacetime with Lorentzian metric $g_{\mu\nu}$, as we assume from now on, the constitutive tensor density is (cf. Post [13], Chapter 9)

$$\chi^{\alpha\beta\gamma\delta} = \sqrt{-g}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) , \tag{8}$$

where $g = \det g_{\mu\nu}$. Since $F_{\mu\nu}$ and $G^{\mu\nu}$ are antisymmetric, in four spacetime dimensions they have six independent components each, corresponding to the E_i, B^i fields, and the D^i and H_i fields, respectively, in space. However, there are different choices for this spatial slicing, yielding eventually different identifications of the analogue model properties. In the following, we consider two important examples.

2.2. Zero Weight Formalism

First, we review the formalism used by Frolov and Shoom [17] in the context of spinoptics, drawing on earlier work by Torres del Castillo and Mercado-Pérez [18]. In this case, the metric of three-space is defined according to

$$\gamma_{ij} = -\frac{g_{ij}}{g_{00}} + a_i a_j , \tag{9}$$

where

$$a_i = -\frac{g_{0i}}{g_{00}} , \tag{10}$$

and the spacetime line element takes the form

$$\mathrm{d}s^2 = -g_{00}\left(-(\mathrm{d}t - a_i\mathrm{d}x^i)^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) . \tag{11}$$

For static spacetimes with $g_{0i} = 0$, the spatial metric γ_{ij} reduces to the optical metric. This optical metric is defined as the Riemannian metric in three-dimensional space whose geodesics are spatial light rays, by Fermat's principle. In the case of stationary metrics with $g_{0i} \neq 0$, spatial light rays obeying Fermat's principle are not geodesics of the Riemannian metric γ_{ij}, but of a Randers–Finsler optical geometry. Indeed, the *Randers data* can be read off immediately from Equation (11) as γ_{ij} and a_i, and can be converted to the data defining the corresponding *Zermelo problem*. For a detailed description of this optical geometry see, e.g., [19]. It may also be noted that the metric in Equation (9) is invariant under both signature change and conformal transformation $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$. Furthermore, given that $g_{ij}g^{jk} = \delta_i^k$ and defining γ^{ij} such that

$$\gamma_{ij}\gamma^{jk} = \delta_i^k \tag{12}$$

as well, one finds that components of the inverse spacetime metric in terms of the spatial metric are given by

$$g^{00} = -\frac{\gamma^{ij}a_i a_j - 1}{g_{00}} , \quad g^{0i} = -\frac{\gamma^{ij}a_j}{g_{00}} , \quad g^{ij} = -\frac{\gamma^{ij}}{g_{00}} . \tag{13}$$

Furthermore, note that,

$$g = \det g_{\mu\nu} = g_{00}\det(g_{ij} - g_{i0}g_{00}^{-1}g_{0j}) = -g_{00}^4\gamma , \tag{14}$$

by applying a standard rule for block matrices. Note also that the three-dimensional Levi–Civita symbols are tensor densities, which are related to the totally antisymmetric tensors according to

$$\epsilon_{ijk} = \frac{e_{ijk}}{\sqrt{\gamma}} , \quad \epsilon^{ijk} = \sqrt{\gamma}e^{ijk} , \tag{15}$$

where $\gamma = \det \gamma_{ij}$. Now, the spatial components of the electromagnetic fields are defined as

$$E_i = F_{i0}\,, \quad \text{a covector field,} \tag{16}$$
$$B^i = \frac{1}{2} e^{ijk} F_{jk}\,, \quad \text{a vector field,} \tag{17}$$
$$D^i = \varepsilon_0 (-g_{00})^2 F^{0i}\,, \quad \text{a vector field,} \tag{18}$$
$$H_i = \frac{1}{2} e_{ijk} H^{jk} = \frac{1}{2} \mu_0^{-1} (-g_{00})^2 e_{ijk} F^{jk}, \quad \text{a covector field.} \tag{19}$$

Spatial duals are defined with respect to γ, thus $H^i = \gamma^{ij} H_j$ and

$$\begin{aligned} H_{ij} &= \gamma_{ia}\gamma_{jb} H^{ab} = \mu_0^{-1} \gamma_{ia}\gamma_{jb} (-g_{00})^2 F^{ab} \\ &= \mu_0^{-1} (F_{ij} + E_i a_j - E_j a_i)\,. \end{aligned} \tag{20}$$

With these definitions, Maxwell's equations take the following form: from Equation (6), one obtains,

$$\partial_i (\sqrt{\gamma} B^i) = 0\,, \tag{21}$$
$$\partial_0 (\ln \sqrt{\gamma}) B^i + \partial_0 B^i + e^{ijk} \nabla_j E_k = 0\,, \tag{22}$$

where ∇_i refers to the covariant derivative with respect to γ_{ij}, and we have used that fact that

$$e^{ijk} \nabla_j E_k = \frac{\epsilon^{ijk}}{\sqrt{\gamma}} \partial_j E_k\,. \tag{23}$$

The other set (Equation (7)) of Maxwell's equations yields,

$$\partial_i (\sqrt{\gamma} D^i) = 0\,, \tag{24}$$
$$\partial_0 (\ln \sqrt{\gamma}) D^i + \partial_0 D^i - e^{ijk} \nabla_j H_k = 0\,, \tag{25}$$

using Equation (14) and

$$\partial_j \left(\sqrt{\gamma} (-g_{00})^2 F^{ij} \right) = \nabla_j \left(\sqrt{\gamma} (-g_{00})^2 F^{ij} \right) = \sqrt{\gamma} \nabla_j \left((-g_{00})^2 F^{ij} \right) = \mu_0 \sqrt{\gamma} e^{ijk} \nabla_j H_k. \tag{26}$$

Notice that, with these definitions, the standard form of the spatial Maxwell's equations is recovered for *stationary* spacetimes where $\partial_0 (\ln \sqrt{\gamma}) = 0$. Moreover, one finds the following constitutive relations,

$$\begin{aligned} E^i &= \gamma^{ij} E_j = \gamma^{ij} g_{j\mu} g_{0\nu} F^{\mu\nu} \\ &= -(g_{00})^2 F^{0i} - g_{00} g_{0j} F^{ij} = \varepsilon_0^{-1} D^i + \mu_0 e^{ijk} a_j H_k \\ &= \varepsilon_0^{-1} (D^i + e^{ijk} a_j H_k)\,, \end{aligned} \tag{27}$$

since, of course, $\varepsilon_0 \mu_0 = 1$ in our choice of units. In addition,

$$H^i = \mu_0^{-1} (B^i - e^{ijk} a_j E_k)\,. \tag{28}$$

To summarise, all spatial electromagnetic fields are defined as vector or covector fields, that is, having zero tensor weight, and the constitutive relations in Equations (27) and (28) are vector field equations. We now consider a somewhat different prescription.

2.3. Unit Weight Formalism

The second spatial formalism reviewed here was used, e.g., by Plebanski [6] and Volkov, Izmest'ev and Skrotskii [20], defining a spatial metric $\tilde{\gamma}_{ij}$ which is conformally related to γ_{ij} of Equation (9),

$$\tilde{\gamma}_{ij} = -g_{00}\gamma_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\,, \tag{29}$$

with its inverse denoted by $\tilde{\gamma}^{ij}$. As with Equation (9), this metric is invariant under sign change and conformal transformation of the spacetime metric. By the same token as above, we find the following components of the inverse metric,

$$g^{00} = \tilde{\gamma}^{ij}a_i a_j + \frac{1}{g_{00}}\,, \quad g^{0i} = \tilde{\gamma}^{ij}a_j\,, \quad g^{ij} = \tilde{\gamma}^{ij}\,, \tag{30}$$

where $a_i = -\frac{g_{0i}}{g_{00}}$ as before, but we also define

$$g^i = -g^{0i} = -\tilde{\gamma}^{ij}a_j\,. \tag{31}$$

Furthermore, note that,

$$g = g_{00}\tilde{\gamma}\,. \tag{32}$$

The electromagnetic field components are now defined as follows,

$$\tilde{E}_i = F_{i0}\,, \quad \text{a covector field,} \tag{33}$$

$$\tilde{B}^i = \frac{1}{2}\epsilon^{ijk}F_{jk}\,, \quad \text{a vector density field,} \tag{34}$$

$$\tilde{D}^i = \varepsilon_0 G^{0i}\,, \quad \text{a vector density field,} \tag{35}$$

$$\tilde{H}_i = \frac{1}{2}\mu_0^{-1}\epsilon_{ijk}G^{jk}\,, \quad \text{a covector field (the } \sqrt{\tilde{\gamma}} \text{ cancel).} \tag{36}$$

Now, given these definitions, Maxwell's Equation (6) becomes

$$\partial_i \tilde{B}^i = 0\,, \tag{37}$$

$$\partial_0 \tilde{B}^i + \epsilon^{ijk}\partial_j \tilde{E}_k = 0\,, \tag{38}$$

and Equation (7) is given by,

$$\partial_i \tilde{D}^i = 0\,, \tag{39}$$

$$\partial_0 \tilde{D}^i - \epsilon^{ijk}\partial_j \tilde{H}_k = 0\,. \tag{40}$$

Comparing with the standard spatial Maxwell's equations as well as the definitions of Section 2.2, it may be noted that divergences here are *not* with respect to the spatial metric $\tilde{\gamma}_{ij}$. Nevertheless, they are appealing for their formal identity with the standard flat space set of Maxwell's equations in vector notation.

Regarding the constitutive relations, one obtains the following relationships whose more detailed derivation can be found in Appendix A,

$$\varepsilon_0^{-1}\tilde{D}^i + \mu_0\epsilon^{ijk}a_j\tilde{H}_k = -\frac{\sqrt{-g}}{g_{00}}\tilde{\gamma}^{ik}\tilde{E}_k\,, \tag{41}$$

and also

$$-\mu_0\frac{\sqrt{-g}}{g_{00}}\tilde{\gamma}^{ia}\tilde{H}_a = -\epsilon^{ijk}a_j\tilde{E}_k + \tilde{B}^i\,. \tag{42}$$

To summarise, unlike the previous case, only some of the spatial electromagnetic fields are defined as tensors (electric and magnetic covector fields) but some as tensor densities (electric displacement and magnetic induction vector density fields). The constitutive relations in Equations (41) and (42) are thus equations of vector density fields of weight $+1$. Thus, we call this the *unit weight* formalism in contrast to the *zero weight* formalism of Section 2.2.

Given the definitions of these two formalisms, we are now ready to state and compare the corresponding gravitational magnetoelectric effects.

3. Gravitational Magnetoelectric Effect

Using the spatial electromagnetic fields, one can rewrite Equations (1) and (2) as follows,

$$D^i = \varepsilon_0 \varepsilon^{ij} E_j + \alpha^{ij} H_j \,, \tag{43}$$
$$B^i = \mu_0 \mu^{ij} H_j + \alpha^{ji} E_j \,, \tag{44}$$

and take this to define the relative permittivity ε^{ij}, the relative permeability μ^{ij}, and the linear magnetoelectric effect α^{ij}. Now, turning first to zero weight formalism of Section 2.2 and comparing Equation (43) with a recast Equation (27), that is,

$$D^i = \varepsilon_0 \gamma^{ij} E_j - e^{ikj} a_k H_j \,, \tag{45}$$

and Equation (44) with a recast Equation (28), that is,

$$B^i = \mu_0 \gamma^{ij} H_j + e^{ikj} a_k E_j \,, \tag{46}$$

one finds, using Equation (13),

$$\varepsilon^{ij} = \mu^{ij} = \gamma^{ij} = -g_{00} g^{ij} \tag{47}$$

or, in other words, electric and magnetic susceptibilities that are *vanishing* and are position-independent,

$$\chi_e^{ij} = 0 = \chi_m^{ij} \,. \tag{48}$$

The magnetoelectric effect can now also be read off, using Equation (10),

$$\alpha^{ij} = e^{ijk} a_k = -e^{ijk} \frac{g_{0k}}{g_{00}} \,, \tag{49}$$

which, in this case, is found to be an antisymmetric tensor with zero tensor weight.

Next, consider the unit weight formalism discussed in Section 2.3, again using tildes to distinguish fields from the first case. Now, by comparing Equation (43) with a rewritten Equation (41), that is,

$$\tilde{D}^i = -\varepsilon_0 \frac{\sqrt{-g}}{g_{00}} \tilde{\gamma}^{ij} \tilde{E}_j - \epsilon^{ikj} a_k \tilde{H}_j \,, \tag{50}$$

where we have used again that $\varepsilon_0 \mu_0 = 1$, and Equation (44) with a rewritten Equation (42), that is,

$$\tilde{B}^i = -\mu_0 \frac{\sqrt{-g}}{g_{00}} \tilde{\gamma}^{ij} \tilde{H}_j + \epsilon^{ikj} a_k \tilde{E}_j \,, \tag{51}$$

we see that, using Equation (30),

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = -\frac{\sqrt{-g}}{g_{00}} \tilde{\gamma}^{ij} = -\sqrt{-g}\, \frac{g^{ij}}{g_{00}} \,. \tag{52}$$

Thus, compared with Equation (47), the medium is still impedance-matched, with relative permittivity and permeability being equal. However, these are now tensor densities of weight +1. Finally, the corresponding magnetoelectric effect is

$$\tilde{\alpha}^{ij} = \epsilon^{ijk} a_k = -\epsilon^{ijk} \frac{g_{0k}}{g_{00}}, \tag{53}$$

which now becomes an antisymmetric tensor density of weight +1, in contrast to Equation (49).

Before moving on to applications, we close this section with some general remarks. First, notice that the relative permittivities and permeabilities defined by Equation (47) as well as Equation (52) are invariant under change of spacetime signature, that is invariant under $g_{\mu\nu} \mapsto -g_{\mu\nu}$. They are also invariant under Weyl rescalings of the metric, that is $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$. Both symmetries also hold for the magnetoelectric effect as defined by Equation (49), but for Equation (53) we only have invariance under signature change.

4. Applications

4.1. Minkowski-Langevin

Our first example is the Minkowski spacetime in Langevin form, that is, in a rotating frame as used to derive the Sagnac effect. Starting from Minkowski in cylindrical polar coordinates,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2, \tag{54}$$

then with $\varphi = \tilde{\varphi} + \omega t$, where ω is an angular speed, one obtains the Langevin form

$$\mathrm{d}s^2 = -\left(1-\rho^2\omega^2\right)\left(\mathrm{d}t - \frac{\rho^2\omega}{1-\rho^2\omega^2}\mathrm{d}\tilde{\varphi}\right)^2 + \mathrm{d}\rho^2 + \frac{\rho^2}{1-\rho^2\omega^2}\mathrm{d}\tilde{\varphi}^2 + \mathrm{d}z^2. \tag{55}$$

In this frame, the zero weight formalism yields

$$\varepsilon^{ij} = \mu^{ij} = \left(1-\rho^2\omega^2\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1-\rho^2\omega^2}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{56}$$

for the relative permittivity and permeability, using Equation (47), and

$$\alpha^{ij} = \rho\omega\left(1-\rho^2\omega^2\right)\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{57}$$

for the magnetoelectric effect, from Equation (49). By contrast, the unit weight formalism gives

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{\rho}{1-\rho^2\omega^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1-\rho^2\omega^2}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

and

$$\tilde{\alpha}^{ij} = \frac{\rho^2\omega}{1-\rho^2\omega^2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{59}$$

by applying Equations (52) and (53), respectively, to Equation (55). These non-vanishing magnetoelectric effects even for a flat spacetime illustrate the importance of the choice of frame. This will be seen even more clearly in the following, by considering four different charts for Schwarzschild.

4.2. Schwarzschild Spacetime

4.2.1. Schwarzschild Coordinates

Since the Schwarzschild metric g_{ij} in Schwarzschild coordinates with line element

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2m}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{60}$$

is manifestly static, $g_{0i} = 0$, we find immediately from Equations (49) and (53) that the gravitational magnetoelectric effect vanishes for both spatial formalisms, $\alpha^{ij} = 0 = \tilde{\alpha}^{ij}$. In the case of the former with zero weight, the relative permittivity and permeability given by Equation (47) are

$$\varepsilon^{ij} = \mu^{ij} = \left(1-\frac{2m}{r}\right)\begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{61}$$

and, in the latter case with unit weight, Equation (52) yields

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{r^2|\sin\theta|}{1-\frac{2m}{r}}\begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{62}$$

Comparison of Equations (61) and (62) shows that the two spatial formalisms give rise to different relative permittivities and permeabilities, even in the asymptotic Minkowski regime.

4.2.2. Null Cone Coordinates

Next, we turn to null cone coordinates, focusing on advanced Eddington–Finkelstein coordinates in which Schwarzschild is, of course, no longer manifestly static. Given the coordinate transformation,

$$\mathrm{d}t = \mathrm{d}v - \frac{\mathrm{d}r}{1-\frac{2m}{r}}, \tag{63}$$

the line element in Equation (60) now takes the form

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{64}$$

and we can apply the two formalisms in this chart, with $x^0 = v$. Then, the zero weight spatial formalism yields the following expressions for the relative permittivity and permeability according to Equation (47),

$$\varepsilon^{ij} = \mu^{ij} = \left(1-\frac{2m}{r}\right)\begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}, \tag{65}$$

and the corresponding gravitational magnetoelectric effect in Equation (49) is

$$\alpha^{ij} = \frac{(1-\frac{2m}{r})^2}{r^2|\sin\theta|} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\frac{2m}{r}} \\ 0 & -\frac{1}{1-\frac{2m}{r}} & 0 \end{bmatrix} . \tag{66}$$

By contrast, for unit weight, Equation (52) implies that the relative permittivity and permeability is

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{r^2|\sin\theta|}{1-\frac{2m}{r}} \begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} , \tag{67}$$

while the gravitational magnetoelectric effect (Equation (53)) now becomes

$$\tilde{\alpha}^{ij} = \frac{1}{1-\frac{2m}{r}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \tag{68}$$

Thus, comparing Equation (61) with Equation (65) and Equation (62) with Equation (67), we conclude that the relative permittivities and permeabilities of the two spatial formalisms are identical for Schwarzschild coordinates and advanced Eddington–Finkelstein coordinates. Moreover, it is interesting that the gravitational magnetoelectric field, which vanishes in Schwarzschild coordinates, is non-vanishing for advanced Eddington–Finkelstein. However, the expressions differ in the two formalisms: for zero weight, Equation (66), the effect vanishes for $r \to \infty$; by contrast, for unit weight, Equation (68), the effect tends to a constant at radial infinity.

We conclude this section with a remark on Kruskal coordinates. Introducing both advanced and retarded null coordinates v, w, then instead of Equation (64) we have

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}v\mathrm{d}w + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) , \tag{69}$$

and recasting this line element with $t = \frac{1}{2}(v+w)$ and $x = \frac{1}{2}(v+w)$, we see that the 2-space $\theta =$ const., $\phi =$ const. is conformally Minkowski. Thus, the metric is diagonal in the chart (t, x, θ, ϕ), and consequently no magnetoelectric effect occurs.

4.2.3. Painlevé–Gullstrand Coordinates

Let us now consider Schwarzschild in Painlevé–Gullstrand coordinates, which are defined for a freely falling observer such that Equation (60) takes the form

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}\tilde{t}^2 + 2\sqrt{\frac{2m}{r}}\mathrm{d}\tilde{t}\mathrm{d}r + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) . \tag{70}$$

Now, on the one hand, the relative permittivities and permeabilities in the zero weight case with Equation (47) yielding

$$\varepsilon^{ij} = \mu^{ij} = \left(1-\frac{2m}{r}\right) \begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} , \tag{71}$$

and the unit weight case with Equation (52) giving

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{r^2|\sin\theta|}{1-\frac{2m}{r}} \begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}, \tag{72}$$

are again the same as for Schwarzschild coordinates and for advanced Eddington–Finkelstein coordinates, respectively. On the other hand, the gravitational magnetoelectric effects for Painlevé–Gullstrand are

$$\alpha^{ij} = \frac{\left(1-\frac{2m}{r}\right)^2}{r^2|\sin\theta|} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\frac{2m}{r}}\sqrt{\frac{2m}{r}} \\ 0 & -\frac{1}{1-\frac{2m}{r}}\sqrt{\frac{2m}{r}} & 0 \end{bmatrix} \tag{73}$$

in the zero weight formalism (Equation (49)), and

$$\tilde{\alpha}^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\frac{2m}{r}}\sqrt{\frac{2m}{r}} \\ 0 & -\frac{1}{1-\frac{2m}{r}}\sqrt{\frac{2m}{r}} & 0 \end{bmatrix} \tag{74}$$

in the unit weight formalism (Equation (53)). While Equations (73) and (74) are again non-zero, unlike in Schwarzschild coordinates, they differ from the corresponding effects in advanced Eddington–Finkelstein coordinates. However, it may be noted that the gravitational magnetoelectric effect vanishes for both spatial formalisms, Equations (73) and (74), in the Minkowski limit $r \to \infty$, unlike the previous case.

4.2.4. Kerr–Schild Coordinates

In Kerr–Schild coordinates, the spacetime metric is expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu\,, \tag{75}$$

where l_μ is null with respect to the Minkowski metric $\eta_{\mu\nu}$. Defining $l^\mu = \eta^{\mu\nu} l_\nu$, the inverse of the metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - l^\mu l^\nu\,, \tag{76}$$

so that l_μ is also null with respect to $g^{\mu\nu}$. It also follows that $\det g_{\mu\nu} = -1$ in Kerr–Schild coordinates. Thus, they are a form of Cartesian coordinates for spacetime in which the metric equals its linear approximation[1].

In fact, if we define $T = v - r$ with v from the advanced Eddington–Finkelstein coordinates, then the Schwarzschild metric in Equation (60) is expressed in Kerr–Schild form in Equation (75) with

$$l_\mu = \sqrt{\frac{2m}{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \tag{77}$$

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1 In the language of Feynmann graphs [21] in this gauge, there is just a single non-vanishing tree graph.

where $r^2 = x^2 + y^2 + z^2$, and, changing to polar coordinates, the line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \frac{2m}{r}(\mathrm{d}T + \mathrm{d}r)^2 \,. \tag{78}$$

One can now derive the relative permittivity and permeability in the zero weight formalism, to find

$$\varepsilon^{ij} = \mu^{ij} = \left(1 - \frac{2m}{r}\right) \begin{bmatrix} 1 - \frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix} , \tag{79}$$

from Equation (47), and in the unit weight formalism,

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{r^2|\sin\theta|}{1 - \frac{2m}{r}} \begin{bmatrix} 1 - \frac{2m}{r} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix} . \tag{80}$$

from Equation (52). The magnetoelectric effects in Equations (49) and (53) are

$$\alpha^{ij} = \frac{(1 - \frac{2m}{r})^2}{r^2|\sin\theta|} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2m}{r} \\ 0 & -\frac{2m}{r} & 0 \end{bmatrix} , \tag{81}$$

and

$$\tilde{\alpha}^{ij} = \frac{1}{1 - \frac{2m}{r}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2m}{r} \\ 0 & -\frac{2m}{r} & 0 \end{bmatrix} , \tag{82}$$

respectively. Once again, we see that the relative permittivities and permeabilities in this chart are identical to their counterparts in the charts discussed before, but the corresponding magnetoelectric effects are different. In the case of Painlevé–Gullstrand coordinates, the magnetoelectric effect may be attributed to the fact that one is using a coordinate system adapted to an ingoing congruence of time-like geodesics, each of zero kinetic energy. By contrast, in the case of Kerr–Schild coordinates, the congruence is null and aligned along the ingoing principal null direction of the Weyl tensor.

4.3. Gravitational Waves

4.3.1. Baldwin–Jeffery-Rosen Coordinates

As final application, we consider linearly polarised plane (pp) gravitational waves, first in Baldwin–Jeffery–Rosen[2] coordinates. These are defined by a chart $x^\mu = (u, v, x^I)$, with $I = 1, 2$, where u, v are null coordinates with respect to the Minkowski metric, such that the spacetime line element is given by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + A_{IJ}(u)\mathrm{d}x^I\mathrm{d}x^J \,. \tag{83}$$

Considering gravitational waves travelling in the x-direction, with $x^I = (y, z)$ say, we can write

$$u = \frac{1}{\sqrt{2}}(x - t) \,, \quad v = \frac{1}{\sqrt{2}}(x + t) \,, \tag{84}$$

[2] Usually referred to as Rosen coordinates, however, cf. [22].

such that Equation (83) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + A_{IJ}(u)\mathrm{d}x^I\mathrm{d}x^J \,, \tag{85}$$

and use this to compute the relative permittivities and permeabilities in the zero weight and unit weight formalisms, that is,

$$\varepsilon^{ij} = \mu^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (A^{-1})^{11} & (A^{-1})^{12} \\ 0 & (A^{-1})^{21} & (A^{-1})^{22} \end{bmatrix} , \tag{86}$$

from Equation (47), where $(A^{-1})^{IJ}$ is the inverse of A_{IJ}, and

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \sqrt{\det A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (A^{-1})^{11} & (A^{-1})^{12} \\ 0 & (A^{-1})^{21} & (A^{-1})^{22} \end{bmatrix} , \tag{87}$$

from Equation (52). It is interesting to note that the corresponding magnetoelectric effects vanish in both formalisms,

$$\alpha^{ij} = 0 \,, \tag{88}$$

$$\tilde{\alpha}^{ij} = 0 \,, \tag{89}$$

again from Equations (49) and (53), although the metric in Equation (83) has a mixed term. (The results in the unit weight formalism, Equations (87) and (89), have already been pointed out in [23].) We now change chart and find a rather different situation.

4.3.2. Brinkmann Coordinates

In Brinkmann coordinates, $x^\mu = (U, V, X^I)$, with $I = 1, 2$, where U, V are null with respect to Minkowski, the line element of a pp gravitational wave is

$$\mathrm{d}s^2 = 2\mathrm{d}U\mathrm{d}V + K_{IJ}(U)X^I X^J \mathrm{d}U^2 + \delta_{IJ}\mathrm{d}X^I\mathrm{d}X^J \,, \tag{90}$$

and K_{IJ} is symmetric, trace-free and an arbitrary function of its argument U. Again, considering gravitational waves in the X-direction, with $X^I = (Y, Z)$, we put

$$U = \frac{1}{\sqrt{2}}(X - T) \,, \quad V = \frac{1}{\sqrt{2}}(X + T) \,, \tag{91}$$

and write

$$K = \frac{1}{2}K_{IJ}(U)X^I X^J \tag{92}$$

for short. Then, Equation (90) becomes

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + K(\mathrm{d}X - \mathrm{d}T)^2 \tag{93}$$

$$= -(1-K)\mathrm{d}T^2 - 2K\mathrm{d}T\mathrm{d}X + (1+K)\mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \,. \tag{94}$$

Comparing Equations (93) and (78), one recognises that in Brinkmann coordinates the metric is of Kerr–Schild form, and hence equal to its own linearised approximation, a classic result by Xanthopoulos [24]. For a discussion of the implications of this fact for graviton stability and vacuum polarisation, as well as the connection with the Carroll group, the reader may wish to consult [23] and references therein.

Here, we note the relative permittivities and permeabilities of a gravitational wave in Brinkmann coordinates, which are found to be

$$\varepsilon^{ij} = \mu^{ij} = (1-K)\begin{bmatrix} 1-K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{95}$$

in the zero weight formalism, and

$$\tilde{\varepsilon}^{ij} = \tilde{\mu}^{ij} = \frac{1}{1-K}\begin{bmatrix} 1-K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{96}$$

in the unit weight formalism, using Equations (47) and (52) as before. The corresponding magnetoelectric effects are given by

$$\alpha^{ij} = (1-K)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -K \\ 0 & K & 0 \end{bmatrix}, \tag{97}$$

and

$$\tilde{\alpha}^{ij} = \frac{1}{1-K}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -K \\ 0 & K & 0 \end{bmatrix}, \tag{98}$$

using Equations (49) and (53), respectively. First, we note that the magnetoelectric effects are non-zero for Brinkmann coordinates, unlike Baldwin–Jeffery–Rosen. Moreover, the relative permittivities and permeabilities in both formalisms reduce to the identity in the Minkowski limit, where $K \to 0$, in keeping with the Kerr–Schild-type property of Brinkmann coordinates. Similarly, the magnetoelectric effects tend to zero in this limit, as expected.

5. Concluding Remarks

The gravitational magnetoelectric effect occurs for metrics with non-zero mixed components g_{0i}, but, since it is spatial, it depends crucially on both the coordinates used, and the definitions of the spatial electromagnetic fields.

Here, we have demonstrated explicitly that, depending on these choices, the gravitational magnetoelectric effect can arise as a tensor (Equation (49)) as well as a tensor density (Equation (53)). Moreover, although the effect is well-known for rotating spacetimes such as the Kerr, we have shown that it is also apparent in coordinate charts where the Schwarzschild spacetime is not manifestly static, such as advanced Eddington–Finkelstein (Equations (66) and (68)), Painlevé–Gullstrand (Equations (73) and (74)), and Kerr–Schild (Equations (81) and (82)) coordinates, and even for Minkowski spacetime in the rotating Langevin frame (that is, Equations (57) and (59)). In addition, for pp gravitational waves, we have seen that the gravitational magnetoelectric effect can be either vanishing, namely for Baldwin–Jeffery–Rosen coordinates (Equations (88) and (89)), or non-vanishing, for Brinkmann coordinates (Equations (97) and (98)). At first glance, this is surprising since there are mixed null terms in the spacetime line elements of both charts.

We hope that these observations on the gravitational magnetoelectric effect will help to provide a different perspective, as well as another basis for concrete computations, regarding the rotation of polarisation under gravity (see, e.g., [25]). Moreover, increasing interest in the optical properties of gravitational waves (cf. [26]) may benefit from this description as an effective optical medium. Finally, if suitable translucent multiferroics could be constructed whose permittivities, permeabilities and magnetoelectric effects mimic their gravitational analogues, they would provide interesting gravitational lens models (on constructing metamaterials, see, e.g., [27]). These could model not only lensing by Kerr black holes but potentially also, as mentioned above, the Schwarzschild lens in non-static slicings.

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Appendix A

The constitutive relations for the zero weight formalism of Section 2.3 can be derived as follows. For the displacement, we have

$$\begin{aligned}\varepsilon_0^{-1}\tilde{D}^i &= \sqrt{-g}g^{0\mu}g^{i\nu}F_{\mu\nu}\\ &= \sqrt{-g}\left(g^{00}g^{ij}F_{0j} + g^{0j}g^{i0}F_{j0} + g^{0j}g^{ik}F_{jk}\right)\\ &= \sqrt{-g}\left(-g^{00}\tilde{\gamma}^{ij}\tilde{E}_j + g^i g^j \tilde{E}_j + \tilde{\gamma}^{ij}\epsilon_{jkl}g^k\tilde{B}^l\right),\end{aligned} \tag{A1}$$

and the magnetic field is given by

$$\begin{aligned}\mu_0\tilde{H}_i &= \frac{1}{2}\sqrt{-g}\epsilon_{ijk}g^{j\mu}g^{k\nu}F_{\mu\nu}\\ &= \frac{1}{2}\sqrt{-g}\epsilon_{ijk}\left(g^{j0}g^{kl}F_{0l} + g^{jl}g^{k0}F_{l0} + g^{jm}g^{kn}F_{mn}\right) = -\sqrt{-g}\epsilon_{ijk}\tilde{\gamma}^{jl}g^k\tilde{E}_l + \frac{1}{2}\sqrt{-g}\epsilon_{ijk}g^{jm}g^{kn}F_{mn}\\ &= -\sqrt{-g}\epsilon_{ijk}\tilde{\gamma}^{jl}g^k\tilde{E}_l - \frac{g_{00}}{\sqrt{-g}}\tilde{\gamma}_{ij}\tilde{B}^j,\end{aligned} \tag{A2}$$

since, using Equation (32),

$$\begin{aligned}-\frac{g_{00}}{\sqrt{-g}}\tilde{\gamma}_{ij}\tilde{B}^j &= -\frac{1}{2}\frac{g_{00}}{\sqrt{-g}}\tilde{\gamma}_{ij}\epsilon^{jmn}F_{mn} = -\frac{1}{2}\frac{g_{00}\sqrt{\tilde{\gamma}}}{\sqrt{-g}}\tilde{\gamma}_{ij}e^{jmn}F_{mn}\\ &= \frac{1}{2}\sqrt{-g_{00}}e_{ijk}\tilde{\gamma}^{jm}\tilde{\gamma}^{kn}F_{mn}\\ &= \frac{1}{2}\underbrace{\sqrt{-g_{00}}\sqrt{\tilde{\gamma}}}_{\sqrt{-g}}\epsilon_{ijk}g^{jm}g^{kn}F_{mn}.\end{aligned} \tag{A3}$$

Thus, combining Equations (A1) and (A2),

$$\begin{aligned}\varepsilon_0^{-1}\tilde{D}^i + \mu_0\epsilon^{ijk}a_j\tilde{H}_k = \sqrt{-g}\tilde{E}_k&\left(\underbrace{-\tilde{\gamma}^{ik}g^j a_j + \tilde{\gamma}^{jk}g^i a_j + g^i g^k - \tilde{\gamma}^{ik}\tilde{\gamma}^{mn}a_m a_n}_{=0} - \frac{\tilde{\gamma}^{ik}}{g_{00}}\right)\\ &- \epsilon^{ijk}\tilde{\gamma}_{kl}a_j\tilde{B}^l\underbrace{\left(-\frac{\sqrt{-g}}{\sqrt{\tilde{\gamma}}} + \frac{g_{00}\sqrt{\tilde{\gamma}}}{\sqrt{-g}}\right)}_{=0},\end{aligned} \tag{A4}$$

again using Equation (32), which yields Equation (41),

$$\varepsilon_0^{-1}\tilde{D}^i + \mu_0\epsilon^{ijk}a_j\tilde{H}_k = -\frac{\sqrt{-g}}{g_{00}}\tilde{\gamma}^{ik}\tilde{E}_k\,. \tag{A5}$$

Moreover, Equation (A2) gives rise to Equation (42),

$$-\mu_0 \frac{\sqrt{-g}}{g_{00}} \tilde{\gamma}^{ia} \tilde{H}_a = \frac{-g}{g_{00}\sqrt{\tilde{\gamma}}} e_{abc} \tilde{\gamma}^{ia} \tilde{\gamma}^{kb} \tilde{\gamma}^{jc} a_j \tilde{E}_k + \tilde{\gamma}^{ia} \tilde{\gamma}_{aj} \tilde{B}^j$$
$$= -\epsilon^{ijk} a_j \tilde{E}_k + \tilde{B}^i \,, \tag{A6}$$

as required.

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