Abstract: In the current work, we study the influence of a finite volume on $2 + 1 \text{SU}(3)$ Polyakov Quark–Meson model (PQM) order parameters, (fluctuations) correlations of conserved charges and the quark–hadron phase boundary. Our study of the PQM model order parameters and the (fluctuations) correlations of conserved charges indicates a sizable shift of the quark–hadron phase boundary to higher values of baryon chemical potential ($\mu_B$) and temperature ($T$) for decreasing the system volume. The detailed study of such effect could have important implications for the extraction of the (fluctuations) correlations of conserved charges of the QCD phase diagram from heavy ion data.

Keywords: chiral lagrangian; quark confinement; quark-gluon plasma

1. Introduction

One of the major aims of the current heavy ion collisions research is to study the properties of the strongly interacting matter created in such collisions theoretically and experimentally. On the experimental level, many facilities have been designed to investigate the strongly interacting matter phase diagram such as Relativistic Heavy Ion Collider (RHIC) program [1], and the Large Hadron Collider (LHC) [2]. Different studies suggest that the strongly interacting matter phase transition from hadronic phase to quark–gluon plasma (QGP) phase be a smooth crossover at low density and high-temperature [3], and first-order phase transition at high density and low temperature [4,5]. Both smooth crossover and first-order phase transitions are expected to be connected by the critical endpoint (CEP), at which the phase transition is expected to be second order. One avenue to map out and study the QCD phase diagram is through the effective models such as the quark–meson (QM) model [6–8], the Nambu–Jona–Lasinio (NJL) model [9], and their Polyakov-loop extended versions [10].

Many studies have been devoted to investigating the QCD phase diagram, higher order moments and the thermodynamics of two- [11,12] and three-quark flavors [13] QM model and even PQM model with different Polyakov-loop potentials. The thermodynamic properties (pressure, the equation of state, the speed of sound, specific heat, trace anomaly, and the bulk viscosity) have been evaluated at finite and vanishing chemical potential [13,14].

The effect of a finite-volume on the strongly interacting matter has been widely studied [15–22]. Those studies include that finite volume has a strong effect on the transition temperature ($T_c$), the location of the critical endpoint and other thermodynamic properties. In PQM model, the transition temperature ($T_c$) shifted to large values as the volume decrease [21] and the location of the critical endpoint is shifted toward large $\mu$ and small $T$ [17,18]. On the other hand, NJL [16] and PNJL [19] indicate that the transition temperature ($T_c$) shift to small values as the volume decrease and the location of the critical endpoint is shifted toward large $\mu$ and small $T$ for $2 + 1$ flavor and toward small $\mu$ and small $T$ for 2 flavors.
In this work, we investigate the effect of the finite volumes on the PQM model order-parameters, phase-transition, and the conserved charges fluctuations and correlations. The present work is organized as follows. In Section 2, we give a brief overview of the PQM model. The PQM model calculations of the order-parameters, thermodynamic properties, and the conserved charges fluctuations and correlations are compared with the LQCD [23,24], and the influence of finite-volume effect on the PQM model conserved-quantities, baryon, charge, strangeness and their correlations are presented in Section 3. We conclude with a summary and an outlook in Section 4.

2. The Polyakov Quark–Meson (PQM) Model

The SU(3) Quark–Meson model with \( N_f = 2 + 1 \) flavor quarks is coupled to Polyakov loop dynamics to formulate the Polyakov Quark–Meson (PQM) model [13]. The related Lagrangian is given as

\[
\mathcal{L} = \mathcal{L}_{\text{chiral}} - \mathcal{U}(\phi, \phi^*, T),
\]

where the chiral part of the Lagrangian, \( \mathcal{L}_{\text{chiral}} = \mathcal{L}_{\text{quark}} + \mathcal{L}_{\text{meson}} \) has \( SU(3)_L \times SU(3)_R \) symmetry [25,26]. The first part provides the fermionic sector, and the second part represents the mesonic contribution; both contributions are extensively discussed in Ref. [14].

The second term in Equation (1), \( \mathcal{U}(\phi, \phi^*, T) \), represents the Polyakov-loop effective potential [27], which is expressed by using the dynamics of the thermal expectation value of a color traced Wilson loop in the temporal direction

\[
\Phi(\vec{x}) = \frac{1}{N_c} \langle \mathcal{P}(\vec{x}) \rangle,
\]

Then, the Polyakov-loop potential and its conjugate read:

\[
\phi = \frac{\langle \text{Tr}_c \mathcal{P} \rangle}{N_c}, \quad \phi^* = \frac{\langle \text{Tr}_c \mathcal{P}^\dagger \rangle}{N_c},
\]

where \( \mathcal{P} \) is the Polyakov loop. This can be represented by a matrix in the color space [27]

\[
\mathcal{P}(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau A_4(\vec{x}, \tau) \right],
\]

where \( \beta = 1/T \) is the inverse temperature and \( A_4 = i A^0 \) is called Polyakov gauge [27,28].

In the case of no quarks and zero quark chemical potential, \( \phi = \phi^* \) and the Polyakov loop is recognized as an order parameter for the deconfinement phase-transition [29]. In the present work, we use Polyakov loop effective potential \( \mathcal{U}(\phi, \phi^*, T) \) as discussed in Refs. [29,30], but with a new dimensionless parameter \( K \) that helps us get a better agreement with the LQCD. Other Polyakov loop potentials [31,32] are also examined in various work. However, the particular selection made for this work does not affect the main conclusions of our work.

\[
\frac{\mathcal{U}(\phi, \phi^*, T)}{T^4} = -\frac{B}{2} \phi \phi^* - \frac{a_1}{6} (\phi^3 + \phi^{*3}) + \frac{a_2}{4} (\phi \phi^*)^2 - K \ln[1 - 6 \phi \phi^* + 4(\phi^3 + \phi^{*3}) - 3(\phi \phi^*)^2],
\]

(6)
where $B$ and $K$ are dimensionless parameters given as:

$$B = b_0 + b_1 \left( \frac{T_0}{T} \right) + b_2 \left( \frac{T_0}{T} \right)^2 + b_3 \left( \frac{T_0}{T} \right)^3, \quad (7)$$

$$K = k_1 \left( \frac{T_0}{T} \right) + k_2 \left( \frac{T_0}{T} \right)^2 + k_3 \left( \frac{T_0}{T} \right)^3 + k_4 \left( \frac{T_0}{T} \right)^4. \quad (8)$$

The mean field approximation is used following Refs. [14,33] to obtain the grand potential as:

$$\Omega(T, \mu_f) = U(\sigma_x, \sigma_y) + U(\phi, \phi^*, T) + \Omega_{\bar{\psi}\psi}(T, \mu_f; \phi, \phi^*), \quad (9)$$

where $\sigma_x$ and $\sigma_y$ are the non-strange and strange chiral condensates, and the first term in Equation (9) is a purely mesonic potential expressed as:

$$U(\sigma_x, \sigma_y) = \frac{m^2}{2} (\sigma_x^2 + \sigma_y^2) - h_\sigma \sigma_x - h_y \sigma_y - \frac{c}{2\sqrt{2}} \sigma_x \sigma_y$$

$$+ \frac{\lambda_1}{2} \sigma_x^2 \sigma_y^2 + \frac{1}{8} (2\lambda_1 + \lambda_2) \sigma_x^4 + \frac{1}{4} (\lambda_1 + \lambda_2) \sigma_y^4. \quad (10)$$

Here, $m^2, h_\sigma, h_y, \lambda_1, \lambda_2$ and $c$ are model parameters, as reported in Ref. [26]. The parameters values used in the current study are listed in Table 1. Different studies [34,35] indicate that extending the PQM model with the vector meson sector will help to accomplish better agreement with LQCD at $T < T_c$. Such correction is not included in this work and shall be discussed in future work.

The third term in Equation (9), $\Omega_{\bar{\psi}\psi}(T, \mu_f; \phi, \phi^*)$, which gives the quark and anti-quark contributions, can be shown as [13],

$$\Omega_{\bar{\psi}\psi}(T, \mu_f; \phi, \phi^*) = -2T \sum_f \int \frac{d^3p}{(2\pi)^3}$$

$$\ln \left[ 1 + 3 \left( \phi + \phi^* e^{-\frac{E_f - \mu_f}{T}} \right) e^{-\frac{E_f - \mu_f}{T}} + e^{-3\frac{E_f - \mu_f}{T}} \right]$$

$$+ \ln \left[ 1 + 3 \left( \phi^* + \phi e^{-\frac{E_f + \mu_f}{T}} \right) e^{-\frac{E_f + \mu_f}{T}} + e^{-3\frac{E_f + \mu_f}{T}} \right]. \quad (11)$$

where $N$ gives the number of the quark flavors, $E_f = \sqrt{p^2 + m_q^2}$ (the index $f$ runs over different quark flavors ($u, d$ and $s$)) is the dispersion relation, energy, of the valence quark and antiquark. Assuming degenerate light quarks, $q \equiv u, d$, then we can give the masses as follows:

$$m_q = g \frac{\sigma_x}{2}, \quad (12)$$

$$m_s = g \frac{\sigma_y}{\sqrt{2}}. \quad (13)$$
The quark chemical potentials $\mu_f$ are related to the baryon ($\mu_B$), strange ($\mu_S$) and charge ($\mu_Q$) chemical potentials via the following transformations [36]:

$$
\begin{align*}
\mu_u &= \frac{\mu_B}{3} + \frac{2\mu_Q}{3}, \\
\mu_d &= \frac{\mu_B}{3} - \frac{\mu_Q}{3}, \\
\mu_s &= \frac{\mu_B}{3} - \frac{\mu_Q}{3} - \mu_S,
\end{align*}
$$

The influences of a finite volume are introduced in the PQM model by following the approximate method illustrated in [19,37] via a lower momentum cut-off $p_{min}[GeV] = \pi/2R[GeV] = \lambda$, where $R$ is the length of a cubic volume. In this analysis, we study a simple situation (lower momentum cut-off). A full implementation of the finite volume would require decent consideration of the effects of the surface and curvature, as well as boundary conditions that are periodic for bosons and anti-periodic for fermions. This full implementation of the boundary conditions leads to an infinite sum over discrete momentum values.

The PQM model has a set of parameters, as discussed in Refs [26,30] and listed in Tables 1 and 2.

**Table 1.** Summary of the QM model parameters employed in the presented calculations.

<table>
<thead>
<tr>
<th>$c$ (MeV)</th>
<th>$\lambda_1$</th>
<th>$m^2$ (MeV$^2$)</th>
<th>$\lambda_2$</th>
<th>$h_x$ (MeV$^3$)</th>
<th>$h_y$ (MeV$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4807.84</td>
<td>1.40</td>
<td>(342.52)$^2$</td>
<td>46.48</td>
<td>(120.73)$^3$</td>
<td>(336.41)$^3$</td>
</tr>
</tbody>
</table>

**Table 2.** Summary of the Polyakov loop potential parameters employed in the presented calculations.

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>7.5</td>
<td>6.75</td>
<td>-1.95</td>
<td>2.625</td>
<td>-7.44</td>
</tr>
<tr>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$k_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>0.25</td>
<td>0.24</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To estimate the model different parameters, $\sigma_x, \sigma_y, \phi$ and $\phi^*$, we minimize the thermodynamic potential, Equation (9), with respect to $\sigma_x, \sigma_y, \phi$ and $\phi^*$, which gives us a set of four equations of motion:

$$
\frac{\partial \Omega}{\partial \sigma_x} = \frac{\partial \Omega}{\partial \sigma_y} = \frac{\partial \Omega}{\partial \phi} = \frac{\partial \Omega}{\partial \phi^*} \bigg|_{min} = 0,
$$

(14)

where $\sigma_x = \sigma_y = \phi = \phi^*$ are the global minimum.

3. Results

In this section, we discuss our PQM model calculations using the parameters summarized in Tables 1 and 2 to illustrate the effect of finite volume on the model order parameters, phase transition and fluctuations/correlations of the conserved charges.
3.1. Order Parameters and Phase Transition

In the following, we present several calculations to illustrate the impacts of the finite volumes on the PQM model order parameters and the chiral phase transition.

Figure 1a–c shows the thermal dependence of the non-strange and strange chiral condensates ($\sigma_x$, $\sigma_y$) and Figure 1d–f shows the Polyakov loops ($\phi$ and $\phi^*$) for different volume selections and different $\mu_B$ values. The upper panels show that both $\sigma_x$ and $\sigma_y$ increase as the system volume is decreased, with larger sensitivity for the non-strange chiral condensates ($\sigma_x$). The lower panels show very little if any, volume dependence for $\phi$ and $\phi^*$ at different $\mu_B$.

As mentioned above, PQM model contains strange and non-strange chiral condensates, which reflect the chiral phase transitions. Using both chiral condensates, we can investigate the finite volume effects on the $SU(3)_{2+1}$ PQM model chiral phase transition via the normalized net-difference condensate $\Delta_{ls}(T)$, as defined in Ref. [38],

$$\Delta_{ls}(T) = \frac{\sigma_x - \frac{h_x}{h_y} \sigma_y}{\sigma_x(0) - \frac{h_x}{h_y} \sigma_y(0)}, \quad (15)$$

where $h_x$ ($h_y$) are non-strange (strange) explicit symmetry breaking parameters.

Figure 2a–c shows the thermal dependence of the normalized net-difference condensate $\Delta_{ls}$ and Figure 2d–f shows the $d\Delta_{ls}/dT$ for different volume selections and different $\mu_B$ values. The upper panels indicate an increase in $\Delta_{ls}(T)$ as the system volume is decreased. The lower panels show that for fixed values of $R$ and $\mu_B$, the $d\Delta_{ls}/dT$ is peaking up at a specific point indicating the phase transition. The peak position is shifted toward lower temperature as the $\mu_B$ value increase.
Figure 2. The same as in Figure 1 but for: the net-difference condensate ($\Delta_{ls}$) (a–c); and $d\Delta_{ls}/dT$ (d–f).

The study of the phase diagram of the PQM model for at fixed volume could be done through mapping out the $\mu_B$ dependence of $\Delta_{ls}(T)$. For a fixed $R$ and $\mu_B$ values, $d\Delta_{ls}/dT$ will peak at a particular point expressing the phase transition. Therefore, the phase diagram can be studied by outlining such points for a wide range of baryon chemical potentials. Figure 3 illustrates the effects of finite volume on the phase diagram. The parameters $T_c$ and $\mu_{Bc}$ represent the transition temperature at $\mu_B = 0.0$ GeV and the transition chemical potential at low temperature, respectively, at $R = \infty$. Our calculations reveal that the PQM model phase diagram in the $(\mu_B, T)$-plane, increases with decreasing the system volume. For the $R = 2.0$ (fm), the $\mu_B$ value at low temperature increased by about 30% and the $T$ value at $\mu_B = 0.0$ GeV increased by about 19% from them values at $R = \infty$ (fm).

Figure 3. Chiral phase diagram for different volumes selections.
3.2. Fluctuations and Correlations of Conserved Charges

The thermodynamics quantities and (diagonal) off-diagonal susceptibilities can be determined by using the thermodynamic pressure as \[ p = -\Omega(T, \mu_f), \] \[ s = dp/dT, \] \[ \epsilon = Ts - p, \]
and
\[ \chi_{ijk}^{BQS} = \frac{\partial^{i+j+k}(p)}{\partial(\mu_B)^i \partial(\mu_Q)^j \partial(\mu_S)^k}. \]

where superscripts \(i, j\) and \(k\) run over integers that represent the derivatives orders. The indexes \(B, Q\) and \(S\) represent the conserved-quantities, baryon, charge, and strangeness, respectively. Equation (19) illustrates the dependence of the (fluctuations) correlations of conserved charges on the temperature, chemical potential, and system volume. The susceptibilities evaluated first by computing the thermodynamic potential at vanishing \(\mu_f\) and then expand the scaled thermodynamic potential in a Taylor series around \(\mu_f/T = 0\).

Before addressing the system volume effect, it is informative to contrast the PQM model thermodynamics quantities and (diagonal) off-diagonal susceptibilities calculations for \((\mu_B = 0\) and \(R = \infty\)), to similar results from LQCD calculations [23,24]. Such comparisons are presented in Figures 4 and 5, which indicate a good agreement between the PQM model and LQCD [23,24]. These comparisons could be improved spatially at low temperature by including the vector mesons sector to the PQM model. We discussed the influence of the finite volume on the model thermodynamics quantities in our previous study [21].

Figure 6 displays the temperature dependence of the normalized conserved-fluctuations, baryon \(\chi_{BB}^{2}\), charge \(\chi_{QQ}^{2}\) and strangeness \(\chi_{SS}^{2}\), respectively. The results are presented for several volume selections at three \(\mu_B\) values, \(\mu_B = 0.0, 0.4\) and \(0.6\) GeV. Our results indicate that the normalized fluctuations decrease with the volume, which quickly trends towards the infinite volume value at high temperature. The non-strange susceptibilities \(\chi_{BB}^{2}\) and \(\chi_{QQ}^{2}\) shows a higher sensitivity to the volume change more than the strange susceptibility \(\chi_{SS}^{2}\). This weak sensitivity to the volume change of the strange quantities could be driving from the large mass of the strange quark.

Figure 7 shows the temperature dependence of the off-diagonal susceptibilities, \(\chi_{BQ}^{2}\), \(\chi_{BS}^{2}\) and \(\chi_{SQ}^{2}\) for several volume selections and different \(\mu_B\) values. The net baryons show a high correlation to the net charge and less correlation to the net strange. Our results indicate that the normalized correlations decrease with the volume which quickly trends towards the infinite volume value at high temperature. In addition, the non-strange correlation \(\chi_{BQ}\) show a higher sensitivity to the volume change.

Figure 4. Comparison of the PQM model pressure density, energy density and entropy to results from LQCD. The comparisons are made for \(\mu_B = 0.0\) GeV; the lines indicate the PQM model calculations and the shaded areas indicate LQCD results from Ref. [23].
In addition, the temperature dependence of the higher order baryon susceptibilities $\chi_n^B$ ($n = 4, 6$ and $8$) for different volume selections at $\mu_B$ values, $\mu_B = 0.0, 0.4$ and $0.6$ GeV are shown in Figure 8. The $n$th-order susceptibilities decrease with the volume selections, and, for $n = 6, 8$ they start to peak around the transition temperature $T_c$. In addition, we observe a stronger oscillation in all higher harmonics as we increase the $\mu_B$ values.

Figure 5. The thermal behavior of PQM model conserved charges fluctuation (a–c); and correlation (d–f) are compared with the same quantities obtained from the LQCD (symbols). The comparisons are made for $\mu_B = 0$. The LQCD conserved charges fluctuation (a–c) and correlation (d–f) are taken from Ref. [24] (Tables III, IV and V).
Figure 6. The thermal behavior of the normalized diagonal susceptibilities, $\chi_{BB}^2$, $\chi_{QQ}^2$ and $\chi_{SS}^2$, for several volume selections at $\mu_B = 0.0, 0.4$ and 0.6 GeV.

Figure 7. The thermal behavior of the normalized off-diagonal susceptibilities, $\chi_{BQ}^{11}$, $\chi_{BS}^{11}$ and $\chi_{QS}^{11}$, for several volume selections at $\mu_B = 0.0, 0.4$ and 0.6 GeV.
4. Conclusions

In this work, we have used the $2 + 1 \, SU(3)$ Polyakov Quark–Meson model (PQM) framework to study the properties of the QCD medium produced at finite volume in heavy ion collisions. This model framework provides several conserved-quantities, baryon, charge, and strangeness, which compare well with those obtained in LQCD calculations for vanishing $\mu_B$. Our calculations indicate that the conserved-quantities ($\chi^{ijk}_{BQS}$) are significantly influenced by finite volume effects. The calculated conserved-quantities decrease with the volume, which quickly trends towards the infinite volume value at high temperature. In addition, the non-strange quantities show a higher sensitivity to the volume change more than the strange ones. Finally, PQM model conserved-quantities suggests that the quark–hadron phase boundary is shifted to higher values of $\mu_B$ and $T$ with decreasing system volume.

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References


4. Ejiri, S. Canonical partition function and finite density phase transition in lattice QCD. *Phys. Rev. D* **2008**, *78*, 074507. [CrossRef]


8. Kovacs, P.; Szep, Z. Influence of the isospin and hypercharge chemical potentials on the location of the CEP in the $\mu_B - T$ phase diagram of the $SU(3)_L \times SU(3)_R$ chiral quark model. *Phys. Rev. D* **2008**, *77*, 065016. [CrossRef]


29. Ratti, C.; Thaler, M.A.; Weise, W. Phases of QCD: Lattice thermodynamics and a field theoretical model. *Phys. Rev. D* 2006, 73, 014019. [CrossRef]